Problem:

Suppose that positive real numbers $a_{i,j}, i, j \in \{1, 2, \ldots, 2020\}$ for each pair $(i, j)$ satisfy $a_{i,j}a_{j,i} = 1$. For each $i = 1, \ldots, 2020$ let $c_i = \sum_{k=1}^{2020} a_{k,i}$. Find the maximal possible value of $\sum_{i=1}^{2020} \frac{1}{c_i}$.

Solution: Answer: 1.

Let $c = \sum_{j=1}^{n} \frac{1}{c_j}$. If $a_{i,j} = 1$ for all $(i, j)$ then $c = 1$. Let us show that $c \leq 1$. By Cauchy-Schwarz inequality we have

\[ \sum_{j=1}^{n} \frac{x_j^2}{a_{ji}} \geq \left( \sum_{j=1}^{n} x_j \right)^2 \sum_{j=1}^{n} a_{ji} \]

for every $i$ and positive real numbers $x_1, \ldots, x_n$. Since $a_{i,j}a_{j,i} = 1$ for every $i$ and $j$, letting $x_j = \frac{1}{c_j}$ in (1) yields

\[ \sum_{j=1}^{n} \frac{a_{ij}}{c_j^2} \geq c^2 \frac{1}{c_i} \]

for every $i$. By adding up the inequality in (2) for every $i$ we obtain

\[ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}}{c_j^2} \geq c^2 \sum_{i=1}^{n} \frac{1}{c_i} = c^3. \]
On the other hand, as

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_{ij}}{c_{j}^{2}} = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{a_{ij}}{c_{j}^{2}} = \sum_{j=1}^{n} \left( \frac{1}{c_{j}^{2}} \sum_{i=1}^{n} a_{ij} \right) = \sum_{j=1}^{n} \left( \frac{1}{c_{j}^{2}} c_{j} \right) = \sum_{j=1}^{n} \frac{1}{c_{j}} = c, \tag{4}
\]

inequality in (3) and equation (4) imply \( c \geq c^3 \). Then, as \( c \) is positive, we see that \( c \leq 1 \).

**Solution 2.** We will prove the inequality by induction over \( n \). For \( n = 2 \), let \( a_{1,2} = a \), then \( \frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{1+a} + \frac{1}{1+1/a} = 1 \). So the inequality holds with equality.

Suppose that the inequality holds for \( n = k \): \( \sum_{i=1}^{k} \frac{1}{c_i} \leq 1 \). We will prove it for \( n = k + 1 \). Note that by Cauchy-Schwarz inequality, for any \( c, a, x \in \mathbb{R} \) we have \( (c + a)\left(\frac{x^2}{c} + \frac{1}{a}\right) \geq (x + 1)^2 \) or

\[
\frac{1}{c + a} \leq \left( \frac{x^2}{c} + \frac{1}{a} \right)(x + 1)^{-2}
\]

Therefore, for any \( x \) we get

\[
\sum_{i=1}^{k} \frac{1}{c_i} + \frac{1}{a_{k+1,i}} \leq \sum_{i=1}^{k} \left( \frac{x^2}{c_i} + \frac{1}{a_{k+1,i}} \right)(x + 1)^{-2} \leq \frac{x^2 + \sum_{i=1}^{k} a_{i,k+1}}{(x + 1)^2}
\]

Now by choosing \( x = \sum_{i=1}^{k} a_{i,k+1} \) we get

\[
\sum_{i=1}^{k+1} \frac{1}{\sum_{j=1}^{k+1} a_{j,i}} = \sum_{i=1}^{k} \frac{1}{c_i + a_{k+1,i}} + \frac{1}{\sum_{j=1}^{k} a_{j,k+1} + a_{k+1,k+1}} \leq \frac{x^2 + x}{(x + 1)^2} + \frac{1}{x + 1} = 1.
\]