Bilkent University
Department of Mathematics

Problem Of The Month

May 2020

Problem:

Let $\mathbb{Z}^+$ be the set of all positive integers. For each function $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ and $\ell \in \mathbb{Z}^+$ let f_ℓ be a composite function $\underbrace{f \circ f \circ \cdots \circ f}_{\ell \text{ times}}$. Find all functions $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying

$$(n-1)^{2020} < \prod_{\ell=1}^{2020} f_\ell(n) < n^{2020} + n^{2019}$$

for each $n \in \mathbb{Z}^+$.

Solution: Answer: The only function is $f(n) = n$. The proof is by induction over n.

If $n = 1$, then we get $0 < \prod_{\ell=1}^{2020} f_\ell(1) < 2$. Therefore, $\prod_{\ell=1}^{2020} f_\ell(1) = 1$. Then, $f(1) = f_1(1) = 1$. Assume that $f(k) = k$ for all $k < n$.

If $f(n) \leq n-1$, then for each $\ell \geq 1$ we get $f_\ell(n) = f(n) \leq n-1$ and $(n-1)^{2020} < \prod_{\ell=1}^{2020} f_\ell(n) \leq (n-1)^{2020}$, a contradiction.

If $f(n) \geq n+1$, we have

$$n^{2020} + n^{2019} > \prod_{\ell=1}^{2020} f_\ell(n) = f(n) \prod_{\ell=2}^{2020} f_\ell(n) \geq (n+1) \prod_{\ell=2}^{2020} f_\ell(n).$$

Thus, we have $n^{2019} > \prod_{\ell=2}^{2020} f_\ell(n)$. Then, we can find l with $2 \leq l \leq 2020$ such that $f_\ell(n) < n$. Let s be the smallest such ℓ. Then, we have $f_{s-1}(n) \geq n$ and $f_s(n) < n$. Let $q = f_{s-1}(n)$. Then $\prod_{i=1}^{2020} f_i(q) \leq (q-1)^{2020}$ and we get a contradiction with the inequality $(n-1)^{2020} < \prod_{\ell=1}^{2020} f_\ell(n)$ at $n = q$:

$$(q-1)^{2020} < \prod_{i=1}^{2020} f_i(q) \leq (q-1)^{2020}.$$

Therefore, $f(n) = n$.