Bilkent University
Department of Mathematics

Problem Of The Month

December 2019

Problem:

Find the maximal value of

$$8abc\,\Big(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\Big)-a-b-c$$

where a, b and c are positive real numbers satisfying $ab+bc+ca \leq 1$.

Solution: The maximal value is $\sqrt{3}$ when $a=b=c=\dfrac{1}{\sqrt{3}}$.

We first observe that $a^2+1 \geq a^2+ab+bc+ca \geq 4a\sqrt{bc}$ where the second inequality follows from AM-GM inequality. Therefore we have $2\sqrt{bc} \geq \dfrac{8abc}{a^2+1}$. Summing this up with similar inequalities involving b^2+1 and c^2+1 yields

$$8abc\,\Big(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\Big) \leq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}).$$

Therefore, in order to show that our expression is not greater than $\sqrt{3}$ we will show that

$$a+b+c+\sqrt{3} \geq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \qquad (1)$$

By $1 \geq ab+bc+ca$ and Cauchy-Schwartz inequality we get

$$\sqrt{3} \geq \sqrt{1+1+1}\sqrt{ab+bc+ca} \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca} \qquad (2)$$

By summing the inequalities $(\sqrt{a}-\sqrt{b})^2 \geq 0, (\sqrt{b}-\sqrt{c})^2 \geq 0, (\sqrt{c}-\sqrt{a})^2 \geq 0$ we get

$$a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca} \qquad (3)$$

Finally, (3) and (2) imply (1). We are done.