Bilkent University
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Problem Of The Month

November 2019

Problem:

In a country between any two cities there is a direct round flight operated by only one of $r \geq 3$ air companies. Suppose that *only* by using of flights of *any* of these r companies it is possible to visit any city from any other city (directly or by using of several flights). Prove that there are three cities such that direct flights between them are operated by three different companies.

Solution:

Let us reformulate the problem in terms of graph theory: Each edge of a complete graph is coloured into one of $r \geq 3$ colors such that for any given colour the graph remains connected after removing of all other coloured edges. Then there exists a three coloured triangle.

Suppose that there are graphs without three coloured triangles. Consider a counterexample G with minimal number of vertices. Let x be any vertex of G, then any colour appears on at least one edge incident to $x \in G$. Let us remove x and all edges incident to x: $T = G - x$. Let us show that edges of T are coloured into at least three colors. Indeed, if edges of T are coloured only by colours 1 and 2, then since G is connected by some other colour 3, all edges incident to x are coloured 3, a contradiction. Now since G is minimal, T is disconnected in some color, say color 0. Suppose that the 0-connected components of T are $T_1, T_2, \ldots$. Note that since no triangle is three coloured, for any pair (i, j) all edges connecting T_i and T_j are identically coloured in some color other than 0. Consider two edges (x, a) and (x, b) coloured into colors 1 and 2 other than 0. If a and b belong to different components, say T_1 and T_2, then consider 0 coloured edges (x, c) and (x, d), where $c \in T_1$ and $d \in T_2$. Since no triangle is three coloured (c, b) is 2 coloured and (a, d) is 1 coloured which contradicts to the fact that all edges connecting T_1 and T_2 are identically coloured. If a and b belong to the same component, say T_1, then consider a 0 coloured edge (x, c), where $c \in T_2$. Now (a, c) should be coloured 1 and (b, c) should be coloured 2 which contradicts to the fact that all edges connecting T_1 and T_2 are identically coloured. Done.