Bilkent University
Department of Mathematics

Problem Of The Month

September 2019

Problem:

Let $p > 2$ be a prime number, $m > 1$ and n be positive integers such that

$$\frac{m^{pn} - 1}{m^n - 1}$$

is a prime number. Show that

$$pn \mid (p-1)^n + 1.$$

Solution:

We first show that n is a power of p. Let $n = p^k t$ where $k \geq 0$ and $t \geq 1$ are integers and p does not divide t. Let $M = m^{p^k}$. By the assumption in the problem $M^{pt} - 1 = (M^t - 1)q$ for some prime number q. Recall that $(M^a - 1, M^b - 1) = M^{(a,b)} - 1$ for all positive integers a and b, and therefore we get $(M^p - 1, M^t - 1) = M - 1$. Since both $M^p - 1$ and $M^t - 1$ divide $M^{pt} - 1$ we see that $\dfrac{(M^p-1)(M^t-1)}{M-1}$ divides $M^{pt} - 1$. In other words, $\dfrac{M^p - 1}{M-1}$ divides $\dfrac{M^{pt}-1}{M^t-1} = q$. Then, $\dfrac{M^p-1}{M-1}$ is either 1 or q. Clearly it is not 1, and hence it is q so we get $t = 1$. Now since p is an odd prime number, the statement $p^{k+1} | (p-1)^{p^k} + 1$ readily follows from the binomial expansion of $(p-1)^{p^k}$.