Bilkent University
Department of Mathematics

Problem Of The Month

June 2019

Problem:

Let $P(x)$ be a non-constant polynomial with real coefficients such that all of its roots are real numbers. Suppose that there exists a polynomial $Q(x)$ with real coefficients such that

$$(P(x))^2 = P(Q(x))$$

for all real numbers x. Determine the maximal possible number of distinct roots of $P(x)$.

Solution: Answer: 1.

Let $P(x) = A(x-r_1)^{d_1}\cdots(x-r_k)^{d_k}$ where $r_1 < \cdots < r_k$. It is easy to see that $Q(x)$ has degree 2 and hence $Q(x) = ax^2+bx+c$ for some real numbers a, b and c. Then the given equality can be written as

$$A^2\prod_{i=1}^{k}(x-r_i)^{2d_i} = A\prod_{i=1}^{k}(ax^2+bx+c-r_i)^{d_i}.$$

Therefore, for each i the roots of $ax^2+bx+c-r_i$ are r_s and r_t for some s and t. On the other hand, the sum of the roots are $-b/a$ for every i. Thus, all the roots of $(P(x))^2$ can be paired in a way that sum of the elements in each pair is the same. Let an r_1 be paired with r_s and an r_k be paired with r_t. Since $r_1 \leq r_t$, $r_s \leq r_k$ and $r_1+r_s = r_k+r_t$, we see that $r_s = r_k$ and $r_t = r_1$. In other words, every r_1 has to be paired with an r_k and every r_k has to be matched with an r_1. Therefore, we obtain that $d_1 = d_k$. By induction it is easy to prove that $d_i = d_{k+1-i}$ for every i and all pairs are of the form $\{r_j, r_{k+1-j}\}$. Consequently, for every m we have that $(c-r_m)/a$ is equal to r_jr_{k+1-j} for some j. However, the numbers of the form r_jr_{k+1-j} can attain at most $\lfloor\frac{k+1}{2}\rfloor$ distinct values and hence we get $k \leq \lfloor\frac{k+1}{2}\rfloor$ which implies $k \leq 1$.

Example: If $P(x) = x$ and $Q(x) = x^2$ then $P(x)^2 = P(Q(x)) = x^2$ and $P(x)$ has only one distinct root.