Problem:

Let $P(x)$ be a non-constant polynomial with real coefficients such that all of its roots are real numbers. Suppose that there exists a polynomial $Q(x)$ with real coefficients such that

$$(P(x))^2 = P(Q(x))$$

for all real numbers $x$. Determine the maximal possible number of distinct roots of $P(x)$.

Solution: Answer: 1.

Let $P(x) = A(x - r_1)^{d_1} \cdots (x - r_k)^{d_k}$ where $r_1 < \cdots < r_k$. It is easy to see that $Q(x)$ has degree 2 and hence $Q(x) = ax^2 + bx + c$ for some real numbers $a, b$ and $c$. Then the given equality can be written as

$$A^2 \prod_{i=1}^{k} (x - r_i)^{2d_i} = A \prod_{i=1}^{k} (ax^2 + bx + c - r_i)^{d_i}.$$  

Therefore, for each $i$ the roots of $ax^2 + bx + c - r_i$ are $r_s$ and $r_t$ for some $s$ and $t$. On the other hand, the sum of the roots are $-b/a$ for every $i$. Thus, all the roots of $(P(x))^2$ can be paired in a way that sum of the elements in each pair is the same. Let an $r_1$ be paired with $r_s$ and an $r_k$ be paired with $r_t$. Since $r_1 \leq r_t$, $r_s \leq r_k$ and $r_1 + r_s = r_k + r_t$, we see that $r_s = r_k$ and $r_t = r_1$. In other words, every $r_1$ has to be paired with an $r_k$ and every $r_k$ has to be matched with an $r_1$. Therefore, we obtain that $d_1 = d_k$. By induction it is easy to prove that $d_i = d_{k+1-i}$ for every $i$ and all pairs are of the form $\{r_j, r_{k+1-j}\}$. Consequently, for every $m$ we have that $(c - r_m)/a$ is equal to $r_jr_{k+1-j}$ for some $j$. However, the numbers of the form $r_jr_{k+1-j}$ can attain at most $\lfloor \frac{k+1}{2} \rfloor$ distinct values and hence we get $k \leq \lfloor \frac{k+1}{2} \rfloor$ which implies $k \leq 1$.

Example: If $P(x) = x$ and $Q(x) = x^2$ then $P(x)^2 = P(Q(x)) = x^2$ and $P(x)$ has only one distinct root.