Bilkent University
Department of Mathematics

Problem Of The Month

March 2019

Problem:

Find the minimal possible value of $ab + bc + ac$ over all positive numbers a, b, c satisfying

$$abc = 1, \quad a + b + c = 5 \quad \text{and}$$

$$(ab + 2a + 2b - 9)(bc + 2b + 2c - 9)(ca + 2c + 2a - 9) \geq 0.$$

Solution: Answer: 5.

Note that

$$ab + 2a + 2b - 9 = \frac{1}{c} + 2(5 - c) - 9 = \frac{1}{c} - 2c + 1 = \frac{1}{c}(2c + 1)(1 - c).$$

The similar formulas are held for $bc+2b+2c-9$ and $ca+2c+2a-9$. Therefore, ($abc = 1$)

$$(ab+2a+2b-9)(bc+2b+2c-9)(ca+2c+2a-9) = (2a+1)(2b+1)(2c+1)(1-a)(1-b)(1-c) \geq 0.$$

Now since $(2a + 1)(2b + 1)(2c + 1) > 0$, we get

$$(1 - a)(1 - b)(1 - c) = -abc - a - b - c + ab + bc + bc + 1 \geq 0.$$

Thus, $ab + bc + ac \geq 5$. The equality holds at $(a, b, c) = (1, 2 - \sqrt{3}, 2 + \sqrt{3})$.