Bilkent University
Department of Mathematics

Problem Of The Month

December 2018

Problem:

In a school consisting of 2019 pupils, any pair of pupils have exactly one common friend. Determine the smallest possible value of the difference between the numbers of friends of the pupil with the most friends and the pupil with the least friends in this school.

Solution: Answer: 2016.

Suppose that A and B are not friends. If X is a friend of A, then by assumption, X and B have exactly one common friend Y. The function that takes each X to the corresponding Y is a bijection between the friends of X and the friends of Y. Indeed, if for two different friends X_1 and X_2 of A the corresponding Y_1 and Y_2 coincide, then the pair (X_1, X_2) has at least two common friends A and Y_1. Therefore, any two pupils who are not friends have the same number of friends.

Let k be a positive integer. Consider the set of pupils with exactly k friends and its complement. Since any pupil in one of these sets must be friends with any pupil in the other set, at least one of these sets has less than 2 pupils in it. It follows that either everyone has the same number of friends or there is a pupil who is friends with everyone.

Suppose that everyone has exactly k friends. On the other hand there are $\binom{2019}{2}$ pairs of pupils in this group. On the other hand, since every pair has exactly one common friend, counting the number of pairs of friends of all pupils must give the same answer. That is, $2019\binom{k}{2} = \binom{2019}{2}$. But this gives $k(k-1) = 2018$, which has no solution.

Hence, the only possibility is that A_1 is friends with everyone, and A_{2i} and A_{2i+1} are friends for $1 \leq i \leq 1009$. The answer is 2018 - 2 = 2016.