Bilkent University
Department of Mathematics

Problem Of The Month

October 2018

Problem:

Let $a_0, a_1, \ldots, a_{100}$ and $b_1, b_2, \ldots, b_{100}$ be two real sequences such that for each $n = 0, 1, \ldots, 99$

$$a_{n+1} = \frac{a_n}{2}, \quad b_{n+1} = \frac{1}{2} - a_n \quad \text{or} \quad a_{n+1} = 2a_n^2, \quad b_{n+1} = a_n$$

holds. Given $a_{100} \leq a_0$, find the maximal possible value of $b_1 + b_2 + \cdots + b_{100}$.

Solution: The answer is 50.

If $a_0 < 0$ then clearly $a_{100} > a_0$. If $a_0 = 0$, then $b_n \in \{0, 1/2\}$ and hence $S = b_1 + b_2 + \cdots + b_{100} \leq 50$. Assume that $a_0 > 0$. In this case all terms of the sequence (a_n) are positive. We have

$$\left(a_n - \frac{a_{n-1}}{2}\right)\left(a_n - 2a_{n-1}^2\right) = 0 \quad \text{and} \quad a_{100} \leq a_0.$$

This equation can be expressed as

$$\frac{a_n}{a_{n-1}} + \frac{a_{n-1}^2}{a_n} = 2a_{n-1} + \frac{1}{2} \tag{1}$$

Side by side sum of the equation (1) for $n = 1, 2, ..., 100$ yields

$$\sum_{n=1}^{100} \frac{a_n}{a_{n-1}} + \sum_{n=1}^{100} \frac{a_{n-1}^2}{a_n} = 2\sum_{n=1}^{100} a_{n-1} + 50 \tag{2}$$

By using of Cauchy-Schwarz inequality we have

$$\sum_{n=1}^{100} \frac{a_{n-1}^2}{a_n} \geq \frac{(a_0 + a_1 + \cdots + a_{99})^2}{a_1 + a_2 + \cdots + a_{100}} \geq a_0 + a_2 + \cdots + a_{99} = \sum_{n=1}^{100} a_{n-1}.$$

Therefore, by (2) we get

$$\sum_{n=1}^{100}\left(\frac{a_n}{a_{n-1}} - a_{n-1}\right) \leq 50.$$

The problem conditions imply that $\dfrac{a_n}{a_{n-1}} - a_{n-1} = b_n$ for all $n = 1, 2, \ldots, 100$. Thus, we get $S \leq 50$.

The equality holds when $a_i = 1/2, \quad i = 0, 1, ..., 100, \quad b_i = 1/2, \quad i = 1, 2, ..., 100.$