Bilkent University
Department of Mathematics

Problem Of The Month

June 2018

Problem:

In the round robin chess tournament organized in a school every two students played one match among themselves. Find the minimal possible number of students in the school if each girl student has at least a wins in matches against boy students and each boy student has at least b wins in matches against girl students.

Solution: The answer is $a+b+\lceil 2\sqrt{ab}\,\rceil$.

Suppose that x girl and y boy students participated in the tournament. Obviously $xy \geq ax+by$ is a necessary condition for the existence of the tournament with given conditions. Let us show that this inequality is also a sufficient condition for the existence of the tournament. Let us consider $x \times y$ score table of the tournament. We put **1** (alternatively **0**) in the intersection of $m-th$ row and $n-th$ column if $m-th$ girl wins (alternatively loses) the match with $n-th$ boy. Let us prove that there is a table in which each row contains at least a entries **1** and each column contains at least b entries **0**. We start with the table where each entry of the first a columns is **1** and each other entry is **0** and step by step move to the table satisfying the conditions. Suppose that $k-th$ column contains less than b **0**'s. Since the total number of **0**'s is not less than by then some $l-th$ column should contain more than b **0**'s. In this case we choose a row such that the intersection of this row with $k-th$ and $l-th$ columns are **1** and **0**, respectively and switch these two entries. After each move some column with **0** shortage gains one **0** and the total number of **1** entries of each row does not change. Therefore after finite number of moves we will get the desired table. Done.

Now let us find the minimal value of $x+y$. The inequality $xy \geq ax+by$ is equivalent to

$$(x-b)(y-a) \geq a \cdot b. \qquad (1)$$

By AM-GM inequality we get

$$(x-b)+(y-a) \geq 2\sqrt{(x-b)(y-a)} \geq 2\sqrt{a \cdot b}$$

Thus, $x + y \geq a + b + \lceil 2\sqrt{ab}\, \rceil$. Let us find x and y satisfying $x + y = a + b + \lceil 2\sqrt{ab}\, \rceil$.

If $\lceil 2\sqrt{ab}\, \rceil = 2k$ is even then $x = b + k$ and $y = a + k$ will satisfy (1). Indeed, $(x - b)(y - a) = k^2 \geq 4ab/4 = ab$.

If $\lceil 2\sqrt{ab}\, \rceil = 2k + 1$ is odd then $x = b + k$ and $y = a + k + 1$ will satisfy (1). Indeed, $(x - b)(y - a) = k(k + 1) \geq 4ab/4 - 1/4$ and since $k(k + 1)$ is an integer $k(k + 1) \geq ab$. Done.