Problem Of The Month

April 2018

Problem:

We say that a group of 18 students is a team if any two students in this group are friends. It is known that in the school any student belongs to at least one team but if any two students end their friendships at least one student does not belong to any team. We say that a team is special if at least one student of the team has no friend outside of this team. Show that any two friends belong to some special team.

Solution:

Let us prove that any two friends A and B belong to some special team. Let us define a longest sequence $S_1, S_2, \ldots, S_m$ of students such that

- for all $1 \leq i < j \leq m$ we have $S_i \neq S_j$
- for each $1 \leq i \leq m$ any team containing $S_i$ also contains $A, B, S_1, S_2, \ldots, S_{i-1}$.

Note that the sequence is well defined since it contains at least one element. Indeed, if $A$ and $B$ end their friendship then some team $S_1$ does not belong to any team. Therefore, any team containing $S_1$ contains both $A$ and $B$ ($S_1$ may coincide with $A$ or $B$).

Let $T$ be any team of $S_m$. Assume that $S_m$ has a friend $S'$ outside of $T$. If $S_m$ and $S'$ end their friendship then there is a student $S''$ which does not belong to any team. Therefore, any team containing $S''$ contains also $S_m$ and consequently contains $A, B, S_1, S_2, \ldots, S_{m-1}$. Note that $S''$ does not coincide with $S_1, S_2, \ldots, S_{m-1}$ since any team of $S_m$ contains $S_1, S_2, \ldots, S_{m-1}$ and any team of $S''$ contains $S'$. Then $S''$ can be added to the sequence $S_1, S_2, \ldots, S_m$. This contradicts the maximality of this sequence. Thus, the team $T$ is special (and it is the only team of $S_m$). Done.