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Problem Of The Month

November 2017

Problem:

Between any two cities of a country consisting of 2017 cities one-way flights are organized so that there is at least one departure from each city. Determine the maximal possible value of k such that no matter how these flights are arranged there are k cities reachable from any city of a country by using of at most two flights.

Solution: Answer: The maximal value of k is 3.

We will write $A \to B$ if the flight between A and B is from A to B. The flight arrangement where $A \to B \to C \to A$ and all other flights incident to A, B, C are directed to A, B, C shows that $k \leq 3$. A city A reachable from any other city by at most two flights will be called *2-reachable*. Let us show that in any flight arrangement there are at least three 2-reachable cities. The problem can be reformulated in term of graph theory: Let G be directed complete graph with n vertices. If the incoming degree $deg_{in}(A)$ of each vertex A is at most $n-2$, then the are at least three 2-reachable vertices. First of all, let us show that a vertex with maximal incoming degree, say A_1 is 2-reachable. Let U_1 be the set of all vertices X with $A_1 \to X$ and W_1 be the set of all vertices Y with $Y \to A_1$. A_1 is reachable from any $Y \in W_1$ directly. If A_1 is not 2-reachable from some $X_0 \in U_1$, then for any $Y \in W_1$ we have $Y \to X_0$ otherwise A_1 is 2-reachable from $X_0 : X_0 \to Y \to A_1$. But then $deg_{in}X_0 \geq deg_{in}(A_1) + 1$, which contradicts the maximality of $deg_{in}A_1$. Now let A_2 be a vertex in U_1 with maximal incoming degree $deg_{in}A_2$ (note that U_1 is not empty). Let U_2 be the set of all vertices X with $A_2 \to X$ and W_2 be the set of all vertices Y with $Y \to A_2$. A_2 is reachable from any $Y \in W_2$ directly. If A_2 is not 2-reachable from some $X_0 \in U_2$, then for any $Y \in W_2$ we have $Y \to X_0$ otherwise A_2 is 2-reachable from $X_0 : X_0 \to Y \to A_2$. But then $deg_{in}X_0 \geq deg_{in}(A_2)+1$, which contradicts the maximality of $deg_{in}A_2$. Now let A_3 be a vertex in U_2 with maximal incoming degree $deg_{in}A_3$ (note that U_2 is not empty). Let U_3 be the set of all vertices X with $A_3 \to X$ and W_3 be the set of all vertices Y with $Y \to A_3$. A_3 is reachable from any $Y \in W_3$ directly. If A_3 is not 2-reachable from some $X_0 \in U_3$, then for any $Y \in W_3$ we have $Y \to X_0$ otherwise A_3 is 2-reachable from $X_0 : X_0 \to Y \to A_3$. But then $deg_{in}X_0 \geq deg_{in}(A_3) + 1$, which contradicts the maximality of $deg_{in}A_3$. Thus, we have found three vertices 2-reachable from any other vertex. A_1, A_2, A_3 are distinct and the proof is completed (if we proceed the same way the new found point A_4 may coincide with A_1).