Bilkent University
Department of Mathematics

Problem Of The Month

October 2017

Problem:

Find all pairs (m, n) of positive integers satisfying

$$m^6 = n^{n+1} + n - 1 \,.$$

Solution: Answer: $(m, n) = (1, 1)$.

If $n = 1$, the only solution is $(m, n) = (1, 1)$. Let $n > 1$.

n is not odd, otherwise since $2n^{\frac{n+1}{2}} + 1 > n^{\frac{n+1}{2}} > n > n - 1 > 0$ we get

$$(n^{\frac{n+1}{2}})^2 < (m^3)^2 = n^{n+1} + n - 1 < (n^{\frac{n+1}{2}} + 1)^2$$

and $(m^3)^2$ is strictly between two consecutive squares.

$n \not\equiv 2 \pmod 3$, otherwise since

$$3n^{\frac{2(n+1)}{3}} + 3n^{\frac{n+1}{3}} + 1 > n^{\frac{2(n+1)}{3}} > n > n - 1 > 0$$

we get

$$(n^{\frac{n+1}{3}})^3 < (m^2)^3 = n^{n+1} + n - 1 < (n^{\frac{n+1}{3}} + 1)^3$$

and $(m^2)^3$ is strictly between two consecutive cubes.

$n \not\equiv 0 \pmod 3$, otherwise

$$n^{n+1} + n - 1 \equiv -1 \equiv (m^3)^2 \pmod 3 \,.$$

Therefore, $n \equiv 4 \pmod 6$. Now

$$m^6 + 3 = n^{n+1} + n + 2 \equiv (-1)^{n+1} + 1 \pmod{n + 1}$$

since n is even we get that $m^6+3 \equiv 0 \pmod{n+1}$. Suppose that prime number p divides $n+1$. Since $n+1 \equiv 5 \pmod 6$ we get $p > 3$ and since $m^6 \equiv -3 \pmod p$ we see that -3 is a quadratic residue modulo p. Since $p > 3$ we get $p \equiv 1 \pmod 3$. Since all prime divisors of $n+1$ are 1 modulo 3 we get that $n+1 \equiv 1 \pmod 3$ and therefore $n \equiv 0 \pmod 3$ which contradicts $n \equiv 4 \pmod 6$. The only solution is $(m, n) = (1, 1)$.