Problem Of The Month

October 2017

Problem:
Find all pairs \((m, n)\) of positive integers satisfying

\[ m^6 = n^{n+1} + n - 1. \]

Solution: Answer: \((m, n) = (1, 1)\).

If \(n = 1\), the only solution is \((m, n) = (1, 1)\). Let \(n > 1\).

\(n\) is not odd, otherwise since \(2n^{\frac{n+1}{2}} + 1 > n^{\frac{n+1}{2}} > n > n - 1 > 0\) we get

\[ (n^{\frac{n+1}{2}})^2 < (m^3)^2 = n^{n+1} + n - 1 < (n^{\frac{n+1}{2}} + 1)^2 \]

and \((m^3)^2\) is strictly between two consecutive squares.

\(n \not\equiv 2 \pmod{3}\), otherwise since

\[ 3n^{\frac{2(n+1)}{3}} + 3n^{\frac{n+1}{3}} + 1 > n^{\frac{2(n+1)}{3}} > n > n - 1 > 0 \]

we get

\[ (n^{\frac{n+1}{3}})^3 < (m^2)^3 = n^{n+1} + n - 1 < (n^{\frac{n+1}{3}} + 1)^3 \]

and \((m^2)^3\) is strictly between two consecutive cubes.

\(n \not\equiv 0 \pmod{3}\), otherwise

\[ n^{n+1} + n - 1 \equiv -1 \equiv (m^3)^2 \pmod{3}. \]

Therefore, \(n \equiv 4 \pmod{6}\). Now

\[ m^6 + 3 = n^{n+1} + n + 2 \equiv (-1)^{n+1} + 1 \pmod{n + 1} \]
since $n$ is even we get that $m^6 + 3 \equiv 0 \pmod{n+1}$. Suppose that prime number $p$ divides $n + 1$. Since $n + 1 \equiv 5 \pmod{6}$ we get $p > 3$ and since $m^6 \equiv -3 \pmod{p}$ we see that $-3$ is a quadratic residue modulo $p$. Since $p > 3$ we get $p \equiv 1 \pmod{3}$. Since all prime divisors of $n + 1$ are 1 modulo 3 we get that $n + 1 \equiv 1 \pmod{3}$ and therefore $n \equiv 0 \pmod{3}$ which contradicts $n \equiv 4 \pmod{6}$. The only solution is $(m, n) = (1, 1)$. 