Problem of the Month

September 2017

Problem:

Let $n$ be a positive integer and $\phi(n)$ be the number of positive integers less than $n$ that are relatively prime to $n$. Find all pairs of positive integers $(m, n)$ satisfying

$$2^n + (n - \phi(n) - 1)! = n^m + 1.$$ 

Solution: Answer: $(m, n) = (2, 2), (2, 4)$.

If $n = 1$ then we get $2 + 1 = 2$. Therefore, $n > 1$.

If $n$ is a prime number, then $\phi(n) = n - 1$ and $2^n = n^m$. Therefore, $m = n = 2$.

If $n = p^2$, where $p$ is a prime number, then $\phi(n) = p^2 - p$ and we get $2^{p^2} + (p - 1)! = p^{2m} + 1$. For $p = 2$ we get $m = 2$. For $p > 2$ we have $(p - 1)! \equiv 2 \pmod{4}$ which yields $p = 3$. But $2^9 + 2 = 514 = 3^{2m} + 1$ has no solution.

In all other cases let $p$ be the smallest prime factor of $n$. Since $1 < p < 2p < \cdots < p^2 < n$, we have $n - 1 - \phi(n) \geq p$ and therefore $p$ divides $(n - \phi(n) - 1)!$. Thus, $p$ divides $2^n - 1$ and $p$ is odd. By Fermat’s little theorem $p$ also divides $2^{p-1} - 1$. Since $n$ and $p - 1$ are coprime for some integers $a$ and $b$ we have $an - b(p - 1) = 1$. Then

$$1 \equiv 2^n \equiv 2^{an} \equiv 2^{1+b(p-1)} \equiv 2 \cdot 2^{b(p-1)} \equiv 2 \pmod{p}$$

This contradiction shows that there is no any other solution.