Bilkent University
Department of Mathematics

Problem Of The Month

September 2017

Problem:

Let n be a positive integer and $\phi(n)$ be the number of positive integers less than n that are relatively prime to n. Find all pairs of positive integers (m, n) satisfying

$$2^n + (n - \phi(n) - 1)! = n^m + 1\,.$$

Solution: Answer: $(m, n) = (2, 2), (2, 4)$.

If $n = 1$ then we get $2 + 1 = 2$. Therefore, $n > 1$.

If n is a prime number, then $\phi(n) = n - 1$ and $2^n = n^m$. Therefore, $m = n = 2$.

If $n = p^2$, where p is a prime number, then $\phi(n) = p^2 - p$ and we get $2^{p^2} + (p-1)! = p^{2m} + 1$. For $p = 2$ we get $m = 2$. For $p > 2$ we have $(p - 1)! \equiv 2 \pmod 4$ which yields $p = 3$. But $2^9 + 2 = 514 = 3^{2m} + 1$ has no solution.

In all other cases let p be the smallest prime factor of n. Since $1 < p < 2p < \cdots < p^2 < n$, we have $n - 1 - \phi(n) \geq p$ and therefore p divides $(n - \phi(n) - 1)!$ Thus, p divides $2^n - 1$ and p is odd. By Fermat's little theorem p also divides $2^{p-1} - 1$. Since n and $p - 1$ are coprime for some integers a and b we have $an - b(p - 1) = 1$. Then

$$1 \equiv 2^n \equiv 2^{an} \equiv 2^{1+b(p-1)} \equiv 2 \cdot 2^{b(p-1)} \equiv 2 \pmod p$$

This contradiction shows that there is no any other solution.