Bilkent University
Department of Mathematics

Problem Of The Month

December 2016

Problem:

Each student in the class has chosen one mathematics and one physics problem out of the set of 20 mathematics and 16 physics problems such that different students choose different pairs of problems. Given that for each student, at least one of the problems chosen by him is chosen by at most one more student, determine the maximum possible number of students in the class.

Solution: The answer is 64.

For $1 \le i \le 20$ and $1 \le j \le 16$ we define $a_{i,j}$ as follows; $a_{i,j} = 1$ if i-th mathematics problem and j-th physics problem are chosen by some student, $a_{i,j} = 0$ otherwise. Now we can reformulate the problem: Find the maximal possible value of the expression $A = \sum_{i=1}^{20} \sum_{j=1}^{16} a_{i,j}$ under the following two conditions:

- $a_{i,j} = 0$ or 1
- if $a_{k,l} = 1$ for some k and l, then at least one of the sums $\sum_{j=1}^{16} a_{k,j}$ and $\sum_{i=1}^{20} a_{i,l}$ does not exceed 2.

First of all, let us show that $A \le 64$. Suppose that $a_{k,l} = 1$. We say that k is *row-good*, if $\sum_{j=1}^{16} a_{k,j} \le 2$; we say that l is *column-good* if $\sum_{i=1}^{20} a_{i,l} \le 2$.

If the total number of row-good values of k is 20, then $A \le 2 \cdot 20 = 40$.
If the total number of column-good values of l is 16, then $A \le 2 \cdot 16 = 32$.
If the total number of row-good values of k is 19, then $A \le 2 \cdot 19 + 16 = 54$.
If the total number of column-good values of l is 15, then $A \le 2 \cdot 16 + 20 = 52$.

Finally, if the total number of row-good values of k is less than or equal to 18 and the total number of 2-good values of l is less than or equal to 14, then the total number of good values is at most 32 and readily $A \leq 2 \cdot 32 = 64$, since the number of nonzero terms of A is less than or equal to twice the number of good values. Thus, $A \leq 64$.

Now let us give an example for $A = 64$. Let $a_{i,j} = 1$ only for all $(i, j) : i \in \{1, 20\}$ or $j \in \{1, 16\}$ except $(1, 1), (1, 16), (20, 1), (20, 16)$. The conditions are readily satisfied and we are done.