**Bilkent University**
**Department of Mathematics**

# PROBLEM OF THE MONTH

December 2015

**Problem:**

Let $a \geq b \geq 0$ be real numbers. Find the area of the region $K \in \mathbb{R}^2$ defined by

$$K = \{(x, y) \in \mathbb{R}^2 : x \geq y \geq 0 \quad \text{and} \quad x^n + y^n \leq a^n + b^n \text{ for all positive integers n}\}.$$

**Solution:**

We will show that

$$(x, y) \in K \quad \Longleftrightarrow \quad x + y \leq a + b \quad \text{and} \quad x \leq a.$$

Firstly, consider a pair $(x, y) \in K$. By taking $n = 0$, we get $x + y \leq a + b$. On the other hand, by letting $n \to \infty$, we find that

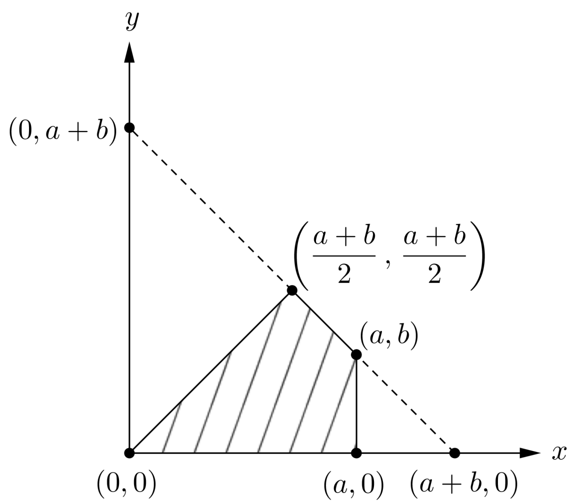
$$x = \max\{x, y\} = \lim_{n\to\infty} \sqrt[n]{\frac{x^n + y^n}{2}} \leq \lim_{n\to\infty} \sqrt[n]{\frac{a^n + b^n}{2}} = \max\{a, b\} = a.$$

Now, assume that $x + y \leq a + b$ and $x \leq a$ for a pair $(x, y)$. We need to prove that $x^n + y^n \leq a^n + b^n$ for all $n \in \mathbb{N}$. This inequality is equivalent to

$$y^n - b^n = (y - b)\left(\sum_{i=0}^{n-1} y^i b^{n-1-i}\right) \leq (a - x)\left(\sum_{i=0}^{n-1} x^i a^{n-1-i}\right) = a^n - x^n$$

which is true since $y - b \leq a - x$ and $0 \leq y \leq x \leq a$, $0 \leq b \leq a$. Therefore, the set $K$ can also be given by

$$K = \{(x, y) \in \mathbb{R}^2 : a \geq x \geq y \geq 0 \quad \text{and} \quad x + y \leq a + b\}.$$




Thus, the area of the region $K$ is equal to

$$\frac{(a+b)^2}{4} - \frac{b^2}{2} = \frac{a^2+2ab-b^2}{4}.$$