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Problem Of The Month

March 2015

Problem:

In each step one can choose two indices $1 \leq k, l \leq 100$ and transform the 100 tuple $(a_1, \ldots, a_k, \ldots, a_l, \ldots, a_{100})$ into the 100 tuple $(a_1, \ldots, \frac{a_k}{2}, \ldots, a_l + \frac{a_k}{2}, \ldots, a_{100})$ if a_k is an even number. We say that a permutation $(a_1, \ldots, a_{100})$ of $(1, 2, \ldots, 100)$ is *good* if starting from $(1, 2, \ldots, 100)$ one can obtain it after finite number of steps. Find the total number of distinct good permutations of $(1, 2, \ldots, 100)$.

Solution: The answer is 100! By the method of mathematical induction we prove that starting from $(1, 2, \ldots, n)$ we can reach any permutation of $(1, 2, \ldots, n)$ for any nonnegative value of n.

1. The case $n = 2$ is clear.
2. Suppose that the statement is true for $n = r - 1$. We will show that starting from $(1, 2, \ldots, r)$ we can reach its arbitrary permutation $(a_1, a_2, \ldots, a_r)$. Let s be an index such that $a_s = r$. First of all, we are going to place r into his desired place by proving that we can reach the permutation

$$(r - s + 1, r - s + 2, \ldots, r - 2, r - 1, r, 1, 2, \ldots, r - s - 1, r - s) \tag{1}$$

Let $T(l)$ be the transformation when the half of the entry a_l has added to a_{l+1} $(a_{r+1} \equiv a_1)$:

$$(a_1, \ldots, a_{l-1}, a_l, a_{l+1}, \ldots, a_k) \to (a_1, \ldots, a_{l-1}, \frac{a_l}{2}, a_{l+1} + \frac{a_l}{2}, \ldots, a_r)$$

It can be readily seen that the series of transformations $T(2), T(3), \ldots, T(r)$ is a cyclic transformation: it shifts $(1, 2, \ldots, r)$ to $(r, 1, 2, \ldots, r-1)$. Similarly, the series $T(3), T(4), \ldots, T(r), T(1)$ will shift $(r, 1, 2, \ldots, r-1)$ to $(r-1, r, 1, 2, \ldots, r-2)$ and $T(4), T(5), \ldots, T(r), T(1), T(2)$ will shift $(r-1, r, 1, 2, \ldots, r-2)$ to $(r-2, r-1, r, 1, 2, \ldots, r-3)$. Thus, after s similar shifts we will get the desired permutation (1). Now the entry r is correctly located, the set of remaining entries is $\{1, 2, \ldots, r-1\}$ and by inductive hypothesis we can get the desired permutation $(a_1, a_2, \ldots, a_r)$. Done.