Bilkent University
Department of Mathematics

Problem Of The Month

October 2014

Problem:

Show that there is a positive integer p for which there exists a sequence of positive integers $\{x_n\}_{n=1}^{\infty}$ such that

• each x_n is a sum of at most p powers of 2: $x_n = 2^{l_1} + 2^{l_2} + \cdots + 2^{l_k}$, where $k \leq p$

and

•• each x_n is divisible by 10^n.

What is the minimal possible value of p?

Solution:

The answer: $p = 2$. First of all, since the power of 2 is not a multiple of 10, $p = 1$ does not satisfy the conditions. Now let us show that the sequence

$$x_n = 2^n + 2^{2\cdot 5^{n-1}+n}; n = 1, 2, \ldots$$

meets the conditions. Since $x_n = 2^n(2^{2\cdot 5^{n-1}} + 1)$ by we prove by induction that $2^{2\cdot 5^{n-1}} + 1$ is divisible by 5^n.

∘ $n = 1 : 2^2 + 1$ is divisible by 5^1.

∘∘ Suppose that $2^{2\cdot 5^{k-1}} + 1 = 4^{5^{k-1}} + 1$ is divisible by 5^k. We have to show that $2^{2\cdot 5^k} + 1$ is divisible by 5^{k+1}. Let $t = 4^{5^{k-1}}$. Then $2^{2\cdot 5^k} + 1 = t^5 + 1 = (t+1)(t^4 - t^3 + t^2 - t + 1)$. By inductive hypothesis the first factor $t + 1$ is divisible by 5^k. Thus, we have to show that $t^4 - t^3 + t^2 - t + 1$ is divisible by 5. Since $t + 1$ is divisible by 5, $t = 5s - 1$ and $t^4 = t^2 = 1$ and $t^3 = t = -1$ in mod 5. Therefore, $t^4 - t^3 + t^2 - t + 1$ is divisible by 5.

The proof is completed.