Problem Of The Month

June 2014

Problem:

Find all triples of positive integers \((a, b, c)\) satisfying \((a^3 + b)(b^3 + a) = 2^c\).

Solution:

Obviously the parities of \(a\) and \(b\) can not be different. Suppose that \(a\) and \(b\) are both even: \(a = 2^l(2a_1 + 1)\) and \(b = 2^m(2b_1 + 1)\), \(l \leq m\). Then after cancelling by \(2^l\) the left hand side of the equation becomes odd. Thus, \(a\) and \(b\) are both odd. If \(a = b\) then the only solution is \(a = b = 1\). Assume \(a > b\). Then \(a^3 + b > b^3 + a\) and since both numbers are powers of \(2\), \(b^3 + a\) divides \(a^3 + b\). Since \(b^3 + a\) also divides \(b^9 + a^3\) we get that \(b^3 + a\) divides their difference \(b^9 - b = b(b^2 - 1)(b^2 + 1)(b^4 + 1)\). Since \(b^3 + a\) is a power of \(2\) and \(b^2 + 1 \geq 2\), \((b^4 + 1) \geq 2\) we get that \(b^3 + a\) divides \(4(b^2 - 1)\) and consequently \(b^3 < 4(b^2 - 1)\). Thus, \(b \leq 3\). If \(b = 1\), then \(a^3 + 1\) and \(a + 1\) both are powers of \(2\). Then \(\frac{a^3 + 1}{a + 1} = a^2 - a + 1\) is also a power of \(2\) which is impossible for odd values of \(a\) exceeding \(1\). If \(b = 3\) then \(b^3 + a\) divides \(4(b^2 - 1)\) yields \(27 + a\) divides \(32\) and \(a = 5\). Thus, solutions are: \((1, 1, 2), (3, 5, 12), (5, 3, 12)\).