Bilkent University
Department of Mathematics

Problem Of The Month

June 2014

Problem:

Find all triples of positive integers (a, b, c) satisfying $(a^3 + b)(b^3 + a) = 2^c$.

Solution:

Obviously the parities of a and b can not be different. Suppose that a and b are both even: $a = 2^l(2a_1 + 1)$ and $b = 2^m(2b_1 + 1)$, $l \leq m$. Then after cancelling by 2^l the left hand side of the equation becomes odd. Thus, a and b are both odd. If $a = b$ then the only solution is $a = b = 1$. Assume $a > b$. Then $a^3 + b > b^3 + a$ and since both numbers are powers of 2, $b^3 + a$ divides $a^3 + b$. Since $b^3 + a$ also divides $b^9 + a^3$ we get that $b^3 + a$ divides their difference $b^9 - b = b(b^2 - 1)(b^2 + 1)(b^4 + 1)$. Since $b^3 + a$ is a power of 2 and $b^2 + 1 \geq 2$, $(b^4 + 1) \geq 2$ we get that $b^3 + a$ divides $4(b^2 - 1)$ and consequently $b^3 < 4(b^2 - 1)$. Thus, $b \leq 3$. If $b = 1$, then $a^3 + 1$ and $a + 1$ both are powers of 2. Then $\frac{a^3+1}{a+1} = a^2 - a + 1$ is also a power of 2 which is impossible for odd values of a exceeding 1. If $b = 3$ then $b^3 + a$ divides $4(b^2 - 1)$ yields $27 + a$ divides 32 and $a = 5$. Thus, solutions are: $(1, 1, 2), (3, 5, 12), (5, 3, 12)$.