**Bilkent University**
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# Problem Of The Month

April 2014

**Problem:**

For a given integer $n \geq 3$, let $S_1, S_2, \dots, S_m$ be distinct subsets of the set $\{1, 2, \dots, n\}$ such that for each $1 \leq i, j \leq m; i \neq j$ the sets $S_i \cap S_j$ contain exactly one element. Determine the maximal possible value of $m$ for each $n$.

**Solution:**

The answer: $m = n$. The $n$ sets $S_1 = \{1\}, S_2 = \{1, 2\}, \dots, S_n = \{1, n\}$ readily meet the conditions. Let us show that $m \leq n$.

Let $S_1, S_2, \dots, S_m$ be distinct subsets satisfying the conditions. For each $S_j$ we define $n$ dimensional vector $v^j = (a_1^j, a_2^j, \dots, a_n^j)$ such that

$$a_i^j = \begin{cases} 1 & if \;\; i \in S_j \\ 0 & if \;\; i \notin S_j \end{cases}$$

Let $|S_i|$ be the number of elements of $S_i$. We show that the vectors $v^1, v^2, \dots, v^m$ are linearly independent and hence $m \leq n$. On the contrary, w.l.o.g. suppose that

$$v^1 = \sum_{i=2}^{m} \alpha_i v^i \tag{1}$$

Let us fix an index $j \neq 1$. The dot product of both sides of (1) and $v^j$ yields

$$1 = \sum_{i=2,3,\ldots,m; i\neq j} \alpha_i + \alpha_j |S_j| = \sum_{i=2}^{m} \alpha_i + \alpha_j |S_j| - \alpha_j \qquad (2)$$

The dot product of both sides of (1) and $v^1$, yields

$$|S_1| = \sum_{i=2}^{m} \alpha_i \qquad (3)$$

Now (2) and (3) give $1 = \alpha_j(|S_j| - 1) + |S_1|$. If $|S_j| = 1$, then obviously $m \leq n$. Otherwise $\alpha_j \leq 0$. Thus, for each $j \neq 1$ $\alpha_j \leq 0$ which contradicts (1).