Problem:

Determine all triples \((p, q, r)\) of nonnegative integers satisfying \(p^3 - q^3 = r! - 18\).

Solution:

The answer: the only solution is \((9, 3, 6)\).

If \(r = 1\) then \(q^3 - p^3 = 17\) and if \(r = 2\) then \(q^3 - p^3 = 16\). Since the sequence of cubes is 1, 8, 27, 64, ... there is no solution in these cases.

If \(r \geq 3\) then \(3|(p^3 - q^3) \Rightarrow 3|(p - q) \Rightarrow 9|(p^3 - q^3) \Rightarrow r \geq 6\).

If \(r \geq 7\) then \(7|(p^3 - q^3 + 18)\), but since \(x^3 \mod 7\) is 0, ±1, there is no solution in these case.

Thus, \(r = 6\) and \(p^3 - q^3 = 702 = 2 \cdot 3^3 \cdot 13\). Since \(p^3 - q^3 = (p - q)((p - q)^2 + 3pq)\), \(p - q\) is divisible by 3. Let \(p - q = 3k\): \(k(3k^2 + pq) = 2 \cdot 3 \cdot 13\). \(k > 2 \Rightarrow k(3k^2 + pq) > 2 \cdot 3 \cdot 13\).

Thus, \(k = 1, 2\). If \(k = 1\) there is no solution since \(pq = 75\) and \(p - q = 3\). If \(k = 2\) then \(pq = 27\) and \(p - q = 6\) yield the only solution \((9, 3, 6)\).