Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2013

Problem:

Find all prime triples (p, q, r) such that $3 \not| p+q+r$ and both $p+q+r$, $pq+qr+rp+3$ are perfect squares. Is there any prime triple (p, q, r) such that $3|p+q+r$ and both $p+q+r$, $pq+qr+rp+3$ are perfect squares?

Solution:

Let us show that one of the primes p, q, r is 2. If all primes p, q, r are odds all possibilities up to permutations are: $(p, q, r) \equiv (1, 1, 1)$, $(1, 1, 3)$, $(1, 3, 3)$, $(3, 3, 3) \pmod 4$. We get a contradiction in the cases $(1, 1, 1)$, $(1, 3, 3)$ since $x^2 = p+q+r \equiv 3 \pmod 4$ and in the cases $(1, 1, 3)$, $(3, 3, 3)$ since $y^2 - 3 = pq+qr+rp \equiv 3 \pmod 4$. Therefore, at least one of p, q, r is equal to 2. W.l.o.g. $p = 2$ and $q \leq r$. Then

$$q + r = x^2 - 2, \quad qr = y^2 - 2x^2 + 1$$

Now if $3 \mid y$ then $(q+2)(r+2) = y^2 + 1 \equiv 1 \pmod 3$. Thus, either $q \equiv r \equiv 2 \pmod 3$ or $q \equiv r \equiv 0 \pmod 3$. But for $q \equiv r \equiv 0 \pmod 3$ we get a contradiction: $x^2 - 2 \equiv 0 \pmod 3$. For $q \equiv r \equiv 2 \pmod 3$ we get $x^2 - 2 \equiv 1 \pmod 3$ and $3 \mid x$, but by assumption $33 \not| x$. Thus, $3 \mid y$ is not possible. Now since $33 \not| x$ we get $x^2 \equiv y^2 \equiv 1 \pmod 3$ and consequently $qr = y^2 - 2x^2 + 1 \equiv 0 \pmod 3$. Thus, $q = 3$. Now $r = x^2 - 5$ and $3r = y^2 - 2x^2 + 1$. Therefore $5r = y^2 - 9 = (y-3)(y+3)$. For $r = 2, 3, 5$ x is not an integer number. Therefore, $r > 5$. Since $y - 3 = 1$ yields no solution $y - 3 = 5$, $r = y + 3$ and $r = 11$. For $x = 4$, $y = 8$ we get $(p, q, r) = (2, 3, 11)$. Therefore, all solutions up to permutations are: $(p, q, r) = (2, 3, 11)$.

$(p, q, r) = (2, 11, 23)$ satisfies the conditions.