Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

May 2013

Problem:

Suppose that for all nonnegative a, b, c satisfying $a+b+c=1$ the inequality

$$\frac{a^2+b^2+c^2+\frac{3}{4}abc}{ab+bc+ca} \geq T$$

is held. What is the maximal possible value of T?

Solution:

The answer: $T=\frac{13}{12}$. First of all, let us show that

$$\frac{a^2+b^2+c^2+\frac{3}{4}abc}{ab+bc+ca} \geq \frac{13}{12}$$

Since $a+b+c=1$ the inequality is equivalent to

$$\frac{(a^2+b^2+c^2)(a+b+c)+\frac{3}{4}abc}{(ab+bc+ca)(a+b+c)} \geq \frac{13}{12}$$

After removing the brackets we get

$$12a^3+12b^3+12c^3 \geq a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+30abc$$

We can prove the last inequality by adding the following seven AG mean inequalities:

$10a^3+10b^3+10c^3 \geq 30\sqrt[3]{a^3b^3c^3}=30abc$

$\frac{1}{3}a^3+\frac{1}{3}a^3+\frac{1}{3}b^3 \geq \sqrt[3]{a^3a^3b^3}=a^2b$

$\frac{1}{3}a^3+\frac{1}{3}a^3+\frac{1}{3}c^3 \geq \sqrt[3]{a^3a^3c^3}=a^2c$

$\frac{1}{3}b^3+\frac{1}{3}b^3+\frac{1}{3}a^3 \geq \sqrt[3]{b^3b^3a^3}=ab^2$

$\frac{1}{3}b^3+\frac{1}{3}b^3+\frac{1}{3}c^3 \geq \sqrt[3]{b^3b^3c^3}=b^2c$

$\frac{1}{3}c^3+\frac{1}{3}c^3+\frac{1}{3}a^3 \geq \sqrt[3]{c^3c^3a^3}=ac^2$

$\frac{1}{3}c^3+\frac{1}{3}c^3+\frac{1}{3}b^3 \geq \sqrt[3]{c^3c^3b^3}=bc^2$

Finally note that at $a=b=c=\frac{1}{3}$ the left side of the inequality is equal to $\frac{13}{12}$. Done.