Bilkent University
Department of Mathematics

Problem Of The Month

January 2013

Problem:

Find the minimum value of the expression

$$\frac{a}{b^3+54}+\frac{b}{c^3+54}+\frac{c}{a^3+54}$$

over all nonnegative real numbers a, b, c satisfying $a+b+c=\dfrac{9}{2}$.

Solution:

The answer: $f(a,b,c)=\dfrac{a}{b^3+54}+\dfrac{b}{c^3+54}+\dfrac{c}{a^3+54}$ takes its minimum value $\dfrac{2}{27}$ at points $(a,b,c)=(0,\frac{3}{2},3),(3,0,\frac{3}{2}),(\frac{3}{2},3,0)$.

Since $a+b+c=\frac{9}{2}$, in order to prove $f(a,b,c)\geq\frac{2}{27}$ we show that

$$\left(\frac{a}{b^3+54}-\frac{a}{54}\right)+\left(\frac{b}{c^3+54}-\frac{b}{54}\right)+\left(\frac{c}{a^3+54}-\frac{c}{54}\right)\geq\frac{2}{27}-\frac{9}{2\cdot 54}$$

or

$$\frac{ab^3}{b^3+54}+\frac{bc^3}{c^3+54}+\frac{ca^3}{a^3+54}\leq\frac{1}{2} \qquad (\dagger)$$

Note that since $\dfrac{b^3+27+27}{3}\geq\sqrt[3]{b^3\cdot 27\cdot 27}=9b$ we get $\dfrac{b}{b^3+54}\leq\dfrac{1}{27}$. Similarly,

$\dfrac{c}{c^3+54} \le \dfrac{1}{27}$ and $\dfrac{a}{a^3+54} \le \dfrac{1}{27}$. Therefore, in order to prove $f(a,b,c) \ge \frac{2}{27}$ it is enough to establish the inequality $ab^2+bc^2+ca^2 \le \frac{27}{2}$.

Now let et (x,y,z) be a permutation of (a,b,c) such that $x \ge y \ge z$. Then by rearrangement inequality

$$ab^2+bc^2+ca^2 \le b\cdot ab+c\cdot bc+a\cdot ca \le x\cdot xy+y\cdot zx+z\cdot yz = y(x+z)^2 - xyz \le y(x+z)^2$$

Finally by AM-GM inequality,

$$y(x+z)^2 = \frac{1}{2}\cdot 2y(x+z)(x+z) \le \frac{1}{2}\cdot \big(\frac{2y+x+z+x+z}{3}\big)^3 = \frac{27}{2}$$

where equality is held at $(a,b,c) = (0,\frac{3}{2},3), (3,0,\frac{3}{2}), (\frac{3}{2},3,0)$ since for equality $2y = x+z$ and $xyz = 0$. Note that in Done.

Remark. In the proof of (†) we have used inequalities $\dfrac{b}{b^3+54} \le \dfrac{1}{27}$, $\dfrac{c}{c^3+54} \le \dfrac{1}{27}$ and $\dfrac{a}{a^3+54} \le \dfrac{1}{27}$ with equalities only at $b=3$, $c=3$ and $a=3$ respectively. Fortunately at $(a,b,c) = (0,\frac{3}{2},3), (3,0,\frac{3}{2}), (\frac{3}{2},3,0)$ two term in (†) vanish and the equality still holds.