**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

March 2012

**Problem:**

We say that a rational number $r \in (0,1)$ is $n$-good if the decimal expansion of $r$ is: $r = 0.r_1 r_2 \ldots, r_n$ and $r_i \neq 9$ for all $i = 1, 2, \ldots, n$. Let $G_n$ be the set off all $n$-good numbers. Find the limit

$$\lim_{n\to\infty} \frac{|G_n|}{S_n}$$

where $|G_n|$ is the number of elements in $G_n$ and $S_n$ is the sum of all elements of $G_n$.

**Solution:**

The answer is $\frac{9}{4}$.

Clearly $|G_n| = 9^n$ and $S_n = \sum \sum_{k=1}^{n} \frac{r_k}{10^k}$, where the first summation is taken over all possible combinations of $(r_1, \ldots, r_n)$ with restriction $r_i \neq 9$. Readily

$$S_n = 9^{n-1} \cdot (0 + 1 + \cdots + 8) \cdot (\frac{1}{10} + \frac{1}{100} + \cdots + \frac{1}{10^n}) = 9^{n-1} \cdot 4 \cdot (1 - \frac{1}{10^n}).$$

Therefore,

$$\lim_{n\to\infty} \frac{|G_n|}{S_n} = \frac{9}{4}.$$