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Problem Of The Month

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Problem:

Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of positive real numbers such that for each $l \in \{2, 3, 4, 5\}$ there are pairwise disjoint subsets $S_1^l, S_2^l, \ldots, S_l^l$ of S satisfying $|S_i^l| = \dfrac{|S|}{l}$; $i = 1, 2, \ldots, l$ ($|A|$ denotes the sum of all elements of the set A). Find the minimal possible value of n.

Solution:

Let us show that $n \geq 9$. Let $\sum_{i=1}^{n} a_i = A$. Suppose that $n \leq 8$. If we take $l = 5$ then $|S_i^5| = \dfrac{A}{5}$ and consequently each $a_i \leq \dfrac{A}{5}$. At least two of the subsets $S_i^5, i = 1, \ldots, 5$ contain one element, so for some s, t we have $a_s = a_t = \dfrac{A}{5}$. Contradiction and $n \geq 9$.
$S = \{1, 2, 4, 5, 7, 8, 10, 11, 12\}$ satisfies conditions:

$l = 2$: $\{4, 5, 10, 11\}, \{1, 2, 7, 8, 12\}$.
$l = 3$: $\{1, 7, 12\}, \{4, 5, 11\}, \{2, 8, 10\}$.
$l = 4$: $\{4, 11\}, \{5, 10\}, \{7, 8\}, \{1, 2, 12\}$.
$l = 5$:$\{1, 11\}, \{2, 10\}, \{4, 8\}, \{5, 7\}, \{12\}$.

Thus, the minimal $n = 9$.

Remark. We can prove $n \geq 9$ also in the case when S is a multiset (some elements of S may coincide) by slightly more detailed analysis. Again suppose that $n \leq 8$. If $l = 4$, then $|S_1^4| = |S_2^4| = |S_l^3| = |S_l^4| = \dfrac{A}{4}$ and therefore each S_i^4 contains at least two elements and $n = 8$. Thus, each S_i^4 contains exactly two elements. Let us show that at least six elements of S are of the the form $\frac{A \cdot k}{5}$ where k is a nonnegative integer.

If there are only two elements of S equal to $\frac{A}{5}$, then there are also at least two elements $\frac{A}{20}$ for getting $\frac{A}{4}$. In order to get $\frac{A}{5}$ in the case $l = 5$ we need at least two elements of the form $\frac{3A}{20}$ to add to elements $\frac{A}{20}$ and in total we have six elements of the form $\frac{A \cdot k}{20}$.

If there are at least three elements of S equal to $\frac{A}{5}$, then there are also at least three elements $\frac{A}{20}$ for getting $\frac{A}{4}$ and in total we have six elements of the form $\frac{A \cdot k}{20}$.

Thus, there are at least six elements of S of the form $\frac{A \cdot k}{20}$ and if we take $l = 3$, at least one of the subsets S_i^3 consists of only elements $\frac{A \cdot k}{20}$. Contradiction since $|S_i^3| = \frac{A}{3}$.