Bilkent University
Department of Mathematics

Problem Of The Month

January 2012

Problem:

Find the maximal possible value of the expression $A = \sum_{i=1}^{2012} \sum_{j=1}^{2012} a_{i,j}$ if the following two conditions are held:

• $a_{i,j} = 0$ or 1

• if for some k and l $a_{k,l} = 1$ then at least one of the sums $\sum_{j=1}^{2012} a_{k,j}$ and $\sum_{i=1}^{2012} a_{i,l}$ does not exceed 2.

Solution:

The answer is 8040. First of all, let us show that $A \leq 8040$. Suppose that $a_{k,l} = 1$: we say that k is 1-good, if $\sum_{j=1}^{2012} a_{k,j} \leq 2$; we say that l is 2-good if $\sum_{i=1}^{2012} a_{i,l} \leq 2$.

If the total number of 1-good values of k is 2012, then $A \leq 2 \cdot 2012 = 4024$.
If the total number of 2-good values of l is 2012, then $A \leq 2 \cdot 2012 = 4024$.
If the total number of 1-good values of k is 2011, then $A \leq 2 \cdot 2011 + 2012 = 6036$.
If the total number of 2-good values of l is 2011, then $A \leq 2 \cdot 2011 + 2012 = 6036$.

Finally, if the total number of 1-good values of k is less than 2011 and the total number of 2-good values of l is less than 2011, then the total number of good values is at most 4020 and readily $A \leq 2 \cdot 4020 = 8040$, since the number of nonzero terms of A is less than twice the number of good values. Done.

Now we give an example for $A = 8040$. Let $a_{i,j} = 1$ only for the following (i, j) pairs:

$i = 1$ and $2 \leq j \leq 2011$;

$i = 2012$ and $2 \leq j \leq 2011$;
$j = 1$ and $2 \leq i \leq 2011$;
$j = 2012$ and $2 \leq i \leq 2011$.

The conditions are readily satisfied and $A = 8040$.