Bilkent University
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PROBLEM OF THE MONTH

June 2010

Problem:

Let $a_1, a_2, \ldots$ be a non-decreasing sequence of natural numbers satisfying $a_{a_k} = 3k$ for any k. Determine the set of all possible values of a_{2010}.

Solution:

Suppose that $a_n = a_m$. Then $a_{a_n} = a_{a_m}$ and $3n = 3m$. Therefore, the sequence is strictly increasing.

Suppose that $a_n \leq n$ for some n. Then $3n = a_{a_n} \leq a_n \leq n$, a contradiction. Thus, $a_n > n$. Note that $a_{3n} = a_{a_{a_n}} = 3a_n$.

Suppose that $a_n \geq 3n$ for some n. Then $3n = a_{a_n} \geq a_{3n} = 3a_n$ or $a_n \leq n$, a contradiction. Therefore, $n < a_n < 3n$. Thus, $a_1 = 2$, $a_2 = a_{a_1} = 3$ and $a_{3^n} = 2 \cdot 3^n$. Now $a_{2 \cdot 3^n} = a_{a_{3^n}} = 3^{n+1}$.

If $3^n < l < 2 \cdot 3^n$, then $a_{3^n} < a_l < a_{2 \cdot 3^n}$. Thus, $2 \cdot 3^n < a_l < 3^{n+1}$. Since the sequence is strictly increasing and $2 \cdot 3^n - 3^n = 3^{n+1} - 2 \cdot 3^n$, we get $a_{3^n + t} = 2 \cdot 3^n + t$.

If $2 \cdot 3^n < l < 3^{n+1}$, then $a_{2 \cdot 3^n} < a_l < a_{3^{n+1}}$. Thus, $3^{n+1} < a_l < 2 \cdot 3^{n+1}$. Let $l = 2 \cdot 3^n + t$. Then $a_{2 \cdot 3^n + t} = 3^{n+1} + s$. Let us determine s. We have $a_{a_{2 \cdot 3^n + t}} = a_{3^{n+1} + s}$. Therefore, $3 \cdot (2 \cdot 3^n + t) = 2 \cdot 3^{n+1} + s$. Thus, $s = 3t$ and $a_{2 \cdot 3^n + t} = 3^{n+1} + 3t$.

All terms of the sequence are uniquely determined. Now $a_{2010} = 3 \cdot a_{670}$ and $a_{670} = a_{2 \cdot 3^5 + 184} = 3^6 + 3 \cdot 184$. Finally, $a_{2010} = 3843$.