Bilkent University
Department of Mathematics

Problem Of The Month

September 2009

Problem:

Let (a_n) be a sequence of real numbers such that $a_1 = 2$ and $a_{n+1} = \dfrac{n}{a_1 + a_2 + \cdots + a_n}$ for each natural number n. Prove that $a_{2009} > 0.9995$.

Solution:

Let us prove that $S_n = a_1 + a_2 + \cdots + a_n > n$ by the method of mathematical induction.

1. If $n = 1$, then $s_1 = 2 > 1$.
2. Suppose that $s_n > n$. Then $s_{n+1} - (n+1) = s_n + a_{n+1} - (n+1) = s_n - (n+1) + \dfrac{n}{s_n} = \dfrac{s_n^2 - (n+1)s_n + n}{s_n} = \dfrac{(s_n - n)(s_n - 1)}{s_n} > 0$. Done.

Now since $a_{n+1} = \dfrac{n}{s_n} < 1$ we get $a_n < 1$ for all $n \geq 2$. Therefore, $s_n = 2 + a_2 + a_3 + \cdots + a_n < n + 1$. Finally, $a_n = \dfrac{n}{s_n} > \dfrac{n}{n+1}$ and $a_{2009} > \dfrac{2009}{2010} > 0.9995$.