Bilkent University
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Problem Of The Month

January 2009

Problem:

Let $f : \mathbb{Z}^+ \times \mathbb{Z} \to \mathbb{Z}$ be a function satisfying the following conditions:

1. $f(0, k) = 1$ if $k = 0, 1$.
2. $f(0, k) = 0$ if $k \neq 0$ and $k \neq 1$.
3. $f(n, k) = f(n-1, k) + f(n-1, k-2n)$ for all $n \geq 1$ and k.

Determine $\displaystyle\sum_{k=0}^{\binom{2009}{2}} f(2008, k)$.

Solution:

A. Let us show that for all $n \geq 0$, $f(n, k) = 0$ if $k < 0$ or $k > n^2 + n + 1$. Proof by induction:

1. $n = 0$: $f(0, k) = 0$ if $k < 0$ or $k > 0^2 + 0 + 1 = 1$.

2. Suppose $f(n-1, k) = 0$ if $k < 0$ or $k > (n-1)^2 + n - 1 + 1 = n^2 - n + 1$. Consider $f(n, k)$. If $k < 0$, then $f(n, k) = f(n-1, k) + f(n-1, k-2n) = 0 + 0 = 0$. If $k > n^2 + n + 1$, then $k - 2n > n^2 - n + 1$ and again $f(n, k) = f(n-1, k) + f(n-1, k-2n) = 0 + 0 = 0$.

B. Let us show that $f(n, k) = f(n, n^2 + n + 1 - k)$ for all $n \geq 0$ and k. Proof by induction:

1. $n = 0: f(0, k) = f(0, 1 - k)$.

2. Suppose $f(n-1, k) = f(n-1, n^2 - n + 1 - k)$. Then $f(n, k) = f(n-1, k) + f(n-1, k-2n) = f(n-1, n^2 - n + 1 - k) + f(n-1, n^2 + n + 1 - k) = f(n-1, n^2 + n + 1 - k - 2n) + f(n-1, n^2 + n + 1 - k) = f(n, n^2 + n + 1 - k)$.

C. Let us show that $\sum_{k=0}^{n^2+n+1} f(n, k) = 2^{n+1}$. Proof by induction:

1. $n = 0: f(0, 0) + f(0, 1) = 2^{0+1}$.

2. Suppose $\sum_{k=0}^{n^2-n+1} f(n-1, k) = 2^n$. Then $\sum_{k=0}^{n^2+n+1} f(n, k) = \sum_{k=0}^{n^2+n+1} f(n-1, k) + \sum_{k=0}^{n^2+n+1} f(n-1, k-2n) = (by A, B) \sum_{k=0}^{n^2-n+1} f(n-1, k) + \sum_{m=0}^{n^2-n+1} f(n-1, m) = 2^n + 2^n = 2^{n+1}$.

Finally, by B and C, $\sum_{k=0}^{\frac{n^2+n}{2}} f(2008, k) = 2^{n+1}/2 = 2^n$.
Therefore, $\sum_{k=0}^{\binom{n}{2}} f(2008, k) = 2^{n-1}$. The answer is 2^{2008}.