Bilkent University
Department of Mathematics

Problem Of The Month

April 2008

Problem:

The sequence $\{x_n\}$ is defined by $x_1 = a$, $x_2 = b$ and $x_n = 2008x_{n-1} - x_{n-2}$ for all $n \geq 2$. Prove that there are positive integers a and b such that for all $n \geq 1$

$$1 + 2006x_n x_{n+1}$$

is a perfect square.

Solution:

We prove prove that if $a = 1$ and $b = 2008$ then $1 + 2006x_n x_{n+1}$ is a perfect square. Let us prove that for all $n \geq 1$

$$x_n^2 + x_{n+1}^2 - 1 = 2008x_n x_{n+1}. \tag{1}$$

Proof by induction:

1. $n = 1 : 1^1 + 2008^2 - 1 = 2008 \cdot 1 \cdot 2008.$

2. Suppose (1) is held for $n = k : x_k^2 + x_{k+1}^2 - 1 = 2008x_k x_{k+1} = x_k \cdot 2008x_{k+1} = x_k \cdot (x_k + x_{k+2}) = x_k^2 + x_k x_{k+2}$.
Then $x_{k+1}^2 - 1 = x_k x_{k+2} = (2008x_{k+1} - x_{k+2})x_{k+2} = 2008x_{k+1}x_{k+2} - x_{k+2}^2$. Therefore, $x_{k+1}^2 + x_{k+2}^2 - 1 = 2008x_{k+1}x_{k+2}$ and (1) is held for $n = k + 1$.

Now we get $1 + 2006x_nx_{n+1} = 1 + 2008x_nx_{n+1} - 2x_nx_{n+1} = x_n^2 + x_{n+1}^2 - 2x_nx_{n+1} = (x_{n+1} - x_n)^2$. Done.