**Problem Of The Month**

February 2007

**Problem:** Let \( P \) be the product of the positive real numbers \( a_1, a_2, \ldots, a_{1024} \). Prove that

\[
\prod_{i=1}^{1024} \left(1 + \frac{1}{a_i^{1024} + a_i^{2048}}\right) \geq \left(1 + \frac{1}{P + P^2}\right)^{1024}
\]

**Solution:**

Let us prove that for arbitrary positive real numbers \( a, b \) the following inequality holds:

\[
\left(1 + \frac{1}{a^2 + a^4}\right) \left(1 + \frac{1}{b^2 + b^4}\right) \geq \left(1 + \frac{1}{ab + a^2b^2}\right)^2 \quad (*)
\]

Let us multiply both sides of (*) by \( a^2b^2(1 + a^2)(1 + b^2)(1 + ab)^2 \):

\[
(1 + a^2 + a^4)(1 + b^2 + b^4)(1 + ab)^2 \geq (1 + a^2)(1 + b^2)(1 + ab + a^2b^2)^2.
\]

After expansion of the brackets (27+24 terms!) and simplifying the inequality (*) has the following form:

\[
a^4 + b^4 - 2a^2b^2 + 2ab^5 + 2a^5b - a^2b^4 - a^4b^2 - 2a^3b^3 + a^2b^6 + a^6b^2 - 2a^4b^4 \geq 0
\]

which is equivalent to

\[
(a^2 - b^2)^2 + ab(2b^4 + 2a^4 - ab^3 - a^3b - 2a^2b^2) + a^2b^2(a^2 - b^2)^2
\]
\[(a^2 - b^2)^2 + ab((a^2 - b^2)^2 + (a - b)^2(a^2 + ab + b^2)) + a^2b^2(a^2 - b^2)^2 \geq 0\]

Since all terms of the last inequality are nonnegative, the inequality (\(\ast\)) is proved.

Now, we prove by induction that for all \(n \geq 1\) and any positive real numbers \(a_1, a_2, \ldots, a_{2^n}\) with product \(P_n\)

\[
\prod_{i=1}^{2^n} \left(1 + \frac{1}{a_i^{2n} + a_i^{2n+1}}\right) \geq \left(1 + \frac{1}{P_n + P_{n+1}^2}\right)^{2^n} \quad (I_n)
\]

In particular, this will give the required inequality when \(n = 10\).

Inequality \((I_1)\) directly follows from \((\ast)\). Assume that \((I_n)\) holds. Then

\[
\prod_{i=1}^{2 \times 2^n} \left(1 + \frac{1}{a_i^{2n} + a_i^{4n+2}}\right) = \prod_{i=1}^{2^n} \left(1 + \frac{1}{a_{2i-1}^{2n} + a_{2i-1}^{4n}}\right) \left(1 + \frac{1}{a_{2i}^{2n} + a_{2i}^{4n}}\right)
\]

\[
\geq \prod_{i=1}^{2^n} \left(1 + \frac{1}{a_{2i-1}^{2n} + a_{2i-1}^{2n+1} a_{2i}^{2n}}\right)^2 \quad (by (\ast)) \geq \left[\left(1 + \frac{1}{\prod_{i=1}^{2n} a_{2i-1} a_{2i} + \left(\prod_{i=1}^{2n} a_{2i-1} a_{2i}\right)^2}\right)\right]^{2^n}
\]

(the latter from \((I_n)\) applied to the numbers \(a_{2i-1} a_{2i} (i = 1, 2, \ldots, 2^n)\)). Thus,

\[
\prod_{i=1}^{2 \times 2^n} \left(1 + \frac{1}{a_i^{2n} + a_i^{4n+2}}\right) \geq \left(1 + \frac{1}{\prod_{i=1}^{2 \times 2^n} a_i + \left(\prod_{i=1}^{2 \times 2^n} a_i\right)^2}\right)^{2 \times 2^n}
\]

The last inequality is \((I_{n+1})\). The proof is completed.

The problem has an alternative solution based on the convexity of the function

\[
\ln(1 + \frac{1}{\exp x + \exp 2x}).
\]