Bilkent University
Department of Mathematics

Problem Of The Month

September 2006

Problem: Suppose that a natural number n is an odd perfect number: n is odd and n is equal to the sum of all its positive divisors (including 1 and excluding n). Prove that n is not divisible by 105.

Note: The existence of any odd perfect number is still unknown.

Solution: Suppose that n is divisible by $105 = 3 \times 5 \times 7$. Consider the prime factorization of n:

$$n = 3^{\alpha_1}\, 5^{\alpha_2}\, 7^{\alpha_3}\, p_4^{\alpha_4} \dots p_k^{\alpha_k},\ \alpha_1 \geq 1, \alpha_2 \geq 1, \alpha_3 \geq 1.$$

Let $S(n)$ be the sum of all positive divisors of n (including 1 and n):

$$S(n) = n(1+\frac{1}{3}+\dots+\frac{1}{3^{\alpha_1}})(1+\frac{1}{5}+\dots+\frac{1}{5^{\alpha_2}})(1+\frac{1}{7}+\dots+\frac{1}{7^{\alpha_3}})(1+\frac{1}{p_4}+\dots+\frac{1}{p_4^{\alpha_4}})\dots(1+\frac{1}{p_k}+\dots+\frac{1}{p_k^{\alpha_k}}).$$

Since n is an odd perfect number $S(n) = 2n$ and $S(n)$ is not divisible by 4. Since all primes in the decomposition of n are odd, α_1 and α_3 are at least 2, otherwise $(1+\frac{1}{3}+\dots+\frac{1}{3^{\alpha_1}}) = \frac{4}{3}$ and $(1+\frac{1}{7}+\dots+\frac{1}{7^{\alpha_3}}) = \frac{8}{7}$ and $S(n)$ is divisible by 4. Finally,

$$2 = \frac{S(n)}{n} \geq (1+\frac{1}{3}+\frac{1}{3^2})(1+\frac{1}{5})(1+\frac{1}{7}+\frac{1}{7^2}) = \frac{13}{9}\frac{6}{5}\frac{57}{49} = \frac{4446}{2205} > 2.$$

Contradiction.