Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

July-August 2006

Problem: Find the minimum of the expression

$$a^4 + b^4 + c^4 - 3abc$$

if a, b, c are real numbers satisfying the conditions: $a \geq 1$ and $a + b + c = 0$.

Solution: The answer is $\frac{3}{8}$.

First of all, we prove two auxiliary inequalities:

1. $bc \leq \frac{a^2}{4}$

Proof: Since $-a = b + c$, it is equivalent to
$bc \leq \dfrac{b^2 + c^2 + 2bc}{4}$ or $(b - c)^2 \geq 0$. Done.

2. $b^4 + c^4 \geq \frac{a^4}{8}$.

Proof: Since $-a = b + c$, it is equivalent to
$8b^4 + 8c^4 \geq b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4$, or
$4b^4 + 4c^4 - 4bc^3 - 4b^3c + 3b^4 + 3c^4 - 6b^2c^2$ or
$4(b^3 - c^3)(b - c) + 3(b^2 - c^2)^2 \geq 0$,
which is true (the signs of $b^3 - c^3$ and $b - c$ are the same). Done.

Due to 1 and 2

$$a^4 + b^4 + c^4 - 3abc \geq a^4 + \frac{a^4}{8} - 3\frac{a^2}{4} = \frac{3}{4}a^2(\frac{3}{2}a^2 - 1) \geq \frac{3}{8}$$

since the function $f(x) = \frac{3}{4}x^2(\frac{3}{2}x^2 - 1)$ is strictly increasing on $[1, \infty)$ interval and it takes its minimum at $x = 1$. If $a = 1$, $b = c = -\frac{1}{2}$ the value of $a^4 + b^4 + c^4 - 3abc$ is exactly $\frac{3}{8}$. Done.