Problem: Find the minimum of the expression

\[ a^4 + b^4 + c^4 - 3abc \]

if \(a, b, c\) are real numbers satisfying the conditions: \(a \geq 1\) and \(a + b + c = 0\).

Solution: The answer is \(\frac{3}{8}\).

First of all, we prove two auxiliary inequalities:

1. \(bc \leq \frac{a^2}{4}\)

   Proof: Since \(-a = b + c\), it is equivalent to
   \[ bc \leq \frac{b^2 + c^2 + 2bc}{4} \text{ or } (b - c)^2 \geq 0. \text{ Done.} \]

2. \(b^4 + c^4 \geq \frac{a^4}{8}\).

   Proof: Since \(-a = b + c\), it is equivalent to
   \[ 8b^4 + 8c^4 \geq b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4, \text{ or} \]
   \[ 4b^4 + 4c^4 - 4bc^3 - 4b^3c + 3b^4 + 3c^4 - 6b^2c^2 \text{ or} \]
   \[ 4(b^3 - c^3)(b - c) + 3(b^2 - c^2)^2 \geq 0, \]
   which is true (the signs of \(b^3 - c^3\) and \(b - c\) are the same). Done.

Due to 1 and 2
\[ a^4 + b^4 + c^4 - 3abc \geq a^4 + \frac{a^4}{8} - 3 \frac{a^2}{4} = \frac{3}{4}a^2(\frac{3}{2}a^2 - 1) \geq \frac{3}{8} \]

since the function \( f(x) = \frac{3}{4}x^2(\frac{3}{2}x^2 - 1) \) is strictly increasing on \([1, \infty)\) interval and it takes its minimum at \( x = 1 \). If \( a = 1, \ b = c = -\frac{1}{2} \) the value of \( a^4 + b^4 + c^4 - 3abc \) is exactly \( \frac{3}{8} \). Done.