Bilkent University
Department of Mathematics

Problem Of The Month

December 2005

Problem: Prove that for all natural numbers $b > a$

$$\frac{ab}{(a,b)} + \frac{(a+1)(b+1)}{(a+1,b+1)} \geq \frac{2ab}{\sqrt{b-a}}$$

where (n, m) denotes the greatest common divisor of natural numbers n and m.

Solution: By applying *Arithmetic Mean - Geometric Mean* inequality we get

$$\frac{ab}{(a,b)} + \frac{(a+1)(b+1)}{(a+1,b+1)} \geq 2\sqrt{\frac{a(a+1)b(b+1)}{(a,b)(a+1,b+1)}} > \frac{2ab}{\sqrt{(a,b)(a+1,b+1)}}$$

In order to complete the solution, now we show that

$$a - b \geq (a,b)(a+1,b+1) \tag{1}$$

Indeed, (a, b) and $(a+1, b+1)$ both divide $b-a$, since the greatest common divisor of two numbers also divides their difference. On the other hand, (a, b) and $(a+1, b+1)$ are relatively prime. Therefore, $(a, b)(a+1, b+1)$ divides $b-a$. Hence (1) is proved.