Bilkent University, Department of Physics

PHYS 453: Nuclear & Particle Physics

Fourth Homework

Due Date: 6 April 2012

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1. Using for the Dirac equation, the Hamiltonian $H = c\gamma^0 \left(\vec{\gamma} \cdot \vec{p} + mc\right)$, where $\vec{p} \equiv \frac{\hbar}{i}\nabla$ is the momentum operator:
   a) Show that orbital angular momentum operator $\vec{L} \equiv \vec{r} \times \vec{p}$ does not commute with H. That means, $\vec{L}$ itself is not conserved. So, there must be some unaccounted angular momentum.
   b) Show that $[H, \vec{J}] = 0$, where $\vec{J}$ is the total angular momentum $\vec{J} = \vec{L} + \vec{S}$, with $\vec{S} = \frac{\hbar}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$.
   c) Show that every bispinor is an eigenstate of $\vec{S}^2$, with eigenvalue $\hbar^2 s(s+1)$, and find s. What then is the spin of a particle described by the Dirac equation?

2. The charge conjugation operator (C) takes a Dirac spinor ψ into the 'charge conjugate' spinor ψ_c, given by
   $$\psi_c = i\gamma^2\psi^*$$
   where γ^2 is the third Dirac matrix. Show that the charge conjugates of plane-wave solutions $u^{(1)}$ and $u^{(2)}$ are $v^{(1)}$ and $v^{(2)}$.

3. Suppose we apply a gauge transformation $A'_\mu = A_\mu + \partial_\mu \lambda$, using the gauge function $\lambda = i\hbar\kappa a e^{-(i/\hbar)p\cdot x}$, where κ is an arbitrary constant and p is the photon momentum.
   a) Show that this λ satisfies, $\Box\lambda = 0$.
   b) Show that this gauge transformation has the effect of modifying ϵ_μ : $\epsilon_\mu \rightarrow \epsilon_\mu + \kappa p_\mu$. This observation leads to a beautifully simple test for the gauge invariance of the QED results: the answer must be unchanged if you replace ϵ_μ by $\epsilon + \kappa p_\mu$.
