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A simple derivation of the bandwidth of a laser oscillator

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A simple derivation of the minimum possible bandwidth for a laser oscillator is given. This derivation stresses the role of spontaneous emission in limiting the bandwidth of the oscillator.

It is well known that light emitted from a laser oscillator may be much narrower in bandwidth than the passband of the laser cavity.\(^1,2\) This occurs because stimulated emission into the lasing mode dominates spontaneous emission into that mode. Calculations show that the laser oscillator bandwidth for a steady-state laser may be in principle as narrow as\(^1,2\)

\[
\Delta \nu_{\text{laser}} \simeq \frac{2\pi c}{P_e} \left( \frac{\Delta \nu}{\Delta N_{\text{th}}} \right)^2 \frac{N_1}{\Delta N_{\text{th}}}, \tag{1}
\]

where \(P_e\) is the power in the external laser beam, \(\nu\) is the laser frequency, \(\Delta \nu\) is the passband of the laser cavity, \(N_1\) is the population per unit volume of the upper laser level, and \(\Delta N_{\text{th}}\) is the threshold population difference per unit volume between the upper and lower laser levels. Equation (1) is often derived using the equivalence of a laser and an LC oscillator (see Ref. 2). As a result of teaching a course on lasers, one of the authors (LWA) found that the textbook derivation left several students uncertain as to the basic physical processes that produce the bandwidth \(\Delta \nu_{\text{laser}}\). The purpose of this paper is to present an alternative, albeit approximate derivation, of the minimum possible bandwidth of a laser. It is hoped that this derivation emphasizes the fundamental role that spontaneous emission plays in limiting the laser bandwidth so that students will have a better physical understanding and hence will find the more formal and more precise derivations understandable.

We assume that oscillation occurs only for one mode of the laser, and we assume that the laser operates in the steady state. Above threshold the population difference per unit volume between the two laser levels is clamped at \(\Delta N = \Delta N_{\text{th}}\), and the number of photons in the oscillating mode is very large.\(^3,4\) If there are on the average \(n\) photons in the mode, then the energy of the photons in the mode is \(n\hbar \nu\) where \(\nu\) is the frequency of the electromagnetic wave in the laser cavity. The energy in the electromagnetic wave is also given by

\[
W = \left\langle \frac{1}{2} \int (cE^2 + \mu H^2) dV \right\rangle, \tag{2}
\]

where \(E\) and \(H\) are, respectively, the electric field and magnetic intensity in the laser cavity and where the brackets indicate a time average. This can be rewritten

\[
W = \frac{1}{2} \int (cE_{\text{RMS}}^2 + \mu H_{\text{RMS}}^2) dV = \int cE_{\text{RMS}}^2 dV = \overline{E_{\text{RMS}}^2}V, \tag{3}
\]

where \(E_{\text{RMS}}\) and \(H_{\text{RMS}}\) are, respectively, the root mean square values of the electric field and magnetic intensity in the laser cavity, where \(E_{\text{RMS}}\) is the average value of \(E_{\text{RMS}}\) over the volume of the laser cavity and where \(V\) is the volume of the laser cavity. Equating the two expressions for the energy in the laser cavity one obtains \(E_{\text{RMS}} = (n\hbar c/V)^{1/2}\). We represent the oscillating electric field in the laser cavity using a phasor diagram as shown in Fig. 1. We arbitrarily take the initial phase of the electric field to be 0 at time \(t = 0\), i.e., we define the origin of time to be such that initially the electric field varies as \(e^{i\omega t}\). In the steady state the number of photons per second resulting from induced emission plus the number of photons per second resulting from spontaneous emission is equal to the total loss of photons per second from the laser cavity. The photons resulting from induced emission have the same phase as the inducing field. On the other hand a spontaneous photon emitted into the oscillating mode is emitted with a phase that is random with respect to the phase of the electromagnetic field in the cavity. It is this phase randomness that causes a minimum bandwidth. In order to estimate the effect of spontaneous emission into the oscillating mode we arbitrarily assume that the spontaneous photon is emitted with a phase difference of \(\pm \pi/2\) from the wave in the cavity. The volume average of the magnitude of the electric field of the single spontaneous photon is \(E_{\text{RMS}} = (\hbar c/V)^{1/2}\). The phasor sum of the electric field in the cavity, \(E_{\text{RMS}}\), plus the electric field of the single photon, \(E_{\text{RMS}}\), is shown in Fig. 1. As can be seen from Fig. 1 the change in phase of the total electromagnetic field in the cavity resulting from the spontaneous emission of one photon into the oscillating mode is given by \(\phi = \pm \pi/2\). Thus after the spontaneous photon is emitted the electromagnetic wave in the cavity has a phase \((\omega t + \pi/2)\). The electromagnetic wave in the cavity now induces photons to be emitted with this new phase. As more spontaneous photons are emitted the phase of the electromagnetic wave in the cavity changes by \(\pi/2\) for each spontaneous photon emitted into the oscillating mode. Because each spontaneous photon is emitted with random phase, the phase of the electromagnetic field in the cavity changes in time in a random fashion sometimes increasing and sometimes decreasing due to spontaneous emission. In our simple model where the difference in phase must be either \(\pi/2\) or \(-\pi/2\) the random walk in phase produces a total change in phase after \(m\) spontaneous photons have been emitted into the oscillating mode of

\[
\Phi = m^{1/2} \phi = (m/n)^{1/2}. \tag{4}
\]

If we denote the upper laser level by 1 and the lower laser level by 2 then the total rate of spontaneous emission is \(N_1 t_{\text{spont}}\), where \(N_1\) is the population per unit volume of level 1, and \(t_{\text{spont}}\) is the spontaneous lifetime for a molecule in level 1 to decay into a molecule in level 2. This emission can go with equal probability into any one of \(pV\) modes where \(p = 8\pi \hbar^2 c^3/\Delta \omega^3\) is the number of modes per unit volume of the laser cavity and where \(\Delta \omega\) is the linewidth of the transition from level 1 to level 2. Thus the rate of emission

into the single oscillating mode is $N_1 / p t_{\text{spont}}$. In a time $t$ there are

$$m = N_1 t / p t_{\text{spont}}$$  \hspace{1cm} (5)

spontaneous photons emitted into the single oscillating mode. In the spirit of Pines and Schlichter\(^4\) we arbitrarily assume that when the electromagnetic wave in the cavity changes in phase by $\Phi = 1$ rad the wave is dephased from the original wave. We take the time required for this as the coherence time of the laser, $t_{\text{coh}}$. This leads to the result

$$t_{\text{coh}} = p t_{\text{spont}} / N_1.$$  \hspace{1cm} (6)

The bandwidth of the laser oscillation is given by

$$\Delta \omega_{\text{laser}} = \frac{1}{N_1 t_{\text{coh}}/p t_{\text{spont}}} = \frac{\Delta N_{\text{th}}}{pt_{\text{spont}}} = \frac{N_1}{\Delta N_{\text{th}}}. \hspace{1cm} (7)$$

The threshold population difference per unit volume is given by\(^1,2\)

$$\Delta N_{\text{th}} = \left( N_1 - \frac{g_1}{g_2} N_2 \right)_{\text{th}} = \frac{8 \pi \nu^2 \Delta \nu}{c^3} \left( \frac{t_{\text{spont}}}{t_c} \right) = \frac{p t_{\text{spont}}}{t_c}.$$  \hspace{1cm} (8)

where $t_c$ is the time for the energy stored in the electromagnetic wave in the cavity to decay to $1/e$ of its original value, where $N_2$ is the population per unit volume of the lower laser level, and where $g_1$ and $g_2$ are the statistical weights of levels 1 and 2, respectively. Combining Eqs. (7) and (8) we obtain

$$\Delta \omega_{\text{laser}} = \frac{1}{N_1 t_{\text{coh}}/p t_{\text{spont}}} = \frac{h \nu}{(n h v / t_c) t_c^2} \left( \frac{N_1}{\Delta N_{\text{th}}} \right). \hspace{1cm} (9)$$

The quantity $nh v / t_c$ is approximately equal to the external power $P_e$ emitted by the laser provided the primary loss mechanism from the cavity is transmission through the end mirror into the external laser beam. The quantity $\delta \omega_c = 1/t_c$ is the width in angular frequency of the individual cavity modes. Thus we have

$$\Delta \omega_{\text{laser}} = \frac{h \nu}{P_e} (\delta \omega_c)^2 \left( \frac{N_1}{\Delta N_{\text{th}}} \right). \hspace{1cm} (10)$$

or

$$\Delta \nu_{\text{laser}} = \frac{2 \pi h \nu}{P_e} (\delta \nu_c)^2 \left( \frac{N_1}{\Delta N_{\text{th}}} \right). \hspace{1cm} (11)$$

Equation (11) is identical to the result obtained by Yariv\(^2\) using a more formal derivation.

It is well known that the laser bandwidth $\Delta \nu_{\text{laser}}$ predicted by this equation is extremely narrow. For example, Yariv\(^2\) calculates that the bandwidth $\Delta \nu_{\text{laser}}$ for a 10-mW He-Ne laser with a loss of 1% per pass due to transmission out of the laser is $\Delta \nu_{\text{laser}} = 5 \times 10^{-4}$ Hz so that

$$\frac{\Delta \nu_{\text{laser}}}{\nu} = \frac{\Delta \nu_{\text{laser}}}{\lambda} = 10^{-18}. \hspace{1cm} (12)$$

Actual lasers are not yet able to achieve this extreme spectral purity.

In summary we have given a simple derivation of the minimum possible bandwidth for a laser oscillator. This bandwidth arises because spontaneous emission of photons into the oscillating mode produces random changes of phase in the electromagnetic field in the laser cavity.

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