# A refinement of Alperin's Conjecture for blocks of the endomorphism algebra of the Sylow permutation module

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**Abstract**

We present a refinement of Alperin's Conjecture involving the blocks of the endomorphism algebra of the permutation module formed by the cosets of a $p$-subgroup. We prove the conjecture in two special cases where every weight module has a simple socle.

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## 1 Statement of the Conjecture

Shortly after proposing his weight conjecture [Alp87], Alperin suggested, in seminars, that one approach towards tackling the conjecture would be to examine the endomorphism algebra $\mathrm{End}_{kG}(kG/S)$ of the permutation $kG$-module $kG/S$. Here, $k$ is an algebraically closed field of prime characteristic $p$ and $S$ is a Sylow $p$-subgroup of a finite group $G$. Naehrig [Nae10] has supplied some empirical evidence to suggest that the simple socle constituents of the regular module of $\mathrm{End}_{kG}(kG/S)$ may serve as an intermediate tool to relate the simple $kG$-modules with the weight $kG$-modules.

Recall, a **weight $kG$-module** is defined to be an indecomposable $kG$-module $W$ such that, letting $P$ be a vertex of $W$, then the $kN_G(P)$-module in Green correspondence with $W$ is the inflation of a simple projective $kN_G(P)/P$-module. The weak form of Alperin's Weight

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Conjecture [Alp87] asserts that the number of isomorphism classes of simple $kG$-modules is equal to the number of isomorphism classes of weight $kG$-modules. The block form of Alperin's Conjecture asserts that, given a block $b$ of $kG$, then the number of isomorphism classes of simple $kGb$-modules is equal to the number of isomorphism classes of weight $kGb$-modules.

By an easy application of Frobenius Reciprocity, every simple $kG$-module occurs in both the socle and the head of $kG/S$. The rationale for the study of $\mathrm{End}_{kG}(kG/S)$ arises from the following observation of Alperin [Alp87, Lemma 1], which tells us that, in particular, every weight $kG$-module occurs in both the socle and the head of $kG/S$.

**Lemma 1.1.** (Alperin.) *Every weight $kG$-module occurs as a direct summand of the Sylow permutation $kG$-module $kG/S$.*

We deem all $kG$-modules to be finite-dimensional. A $kG$-module $L$ is said to be **connected** provided $\mathrm{End}_{kG}(L)$ has a unique block. It is easy to see that a direct summand $L$ of a $kG$-module $M$ is maximal among the connected direct summands of $M$ if and only if $L = eM$ for some block $e$ of $\mathrm{End}_{kG}(M)$. When these equivalent conditions hold, we call $L$ a **proper component** of $M$. Plainly, any $kG$-module is the direct sum of its proper components.

We say that a $kG$-module $L$ **lies in** a $kG$-module $M$, written $L \dashv M$, provided $L$ is isomorphic to the image of a $kG$-endomorphism of a direct sum of finitely many copies of $M$. This is equivalent to the condition that there exists a direct sum $M'$ of finitely many copies of $M$ such that $L$ is isomorphic to a submodule of $M'$ and $L$ is isomorphic to a quotient module of $M'$. We say that $M$ is **accordant** provided the number of isomorphism classes of simple $kG$-modules lying in $M$ is equal to the number of isomorphism classes of weight $kG$-modules lying in $M$.

Using Lemma 1.1, it is not hard to see that, for any $p$-subgroup $P$ of $G$, the weak form of Alperin's Conjecture holds for $kG$ if and only if the permutation $kG$-module $kG/P$ is accordant.

**Conjecture 1.2.** *For any $p$-subgroup $P$ of $G$, every proper component of $kG/P$ is accordant.*

The next three remarks are very easy and we omit the proofs.

**Remark 1.3.** *Given a connected $kG$-module $L$ lying in a $kG$-module $M$, then $L$ lies in a unique proper component of $M$.*

**Remark 1.4.** *Let $U$ and $V$ be connected $kG$-modules lying in a $kG$-module $M$. Then $U$ and $V$ lie in the same proper component of $M$ if and only if there exist connected $kG$-modules $W_0$, ..., $W_r$ lying in $M$ such that $W_0 \cong U$ and $W_r \cong V$ and for each $1 \leq i \leq r$, there exists a non-zero $kG$-map $W_{i-1} \to W_i$ or $W_{i-1} \leftarrow W_i$.*

**Remark 1.5.** *Let $L$ and $M$ be $kG$-modules such that $L \dashv M$. Let $U$ and $V$ be connected $kG$-modules lying in $L$. Then $U$ and $V$ lie in $M$. If $U$ and $V$ lie in the same proper component of $L$, then $U$ and $V$ lie in the same proper component of $M$.*

In the special case where $P$ is trivial, Conjecture 1.2 is equivalent to the block form of Alperin's Conjecture. So the next result can be interpreted as saying that Conjecture 1.2 is a refinement of Alperin's Conjecture.

**Proposition 1.6.** *Let $P$ and $Q$ be $p$-subgroups of $G$ with $P \leq Q$. If every proper component of $kG/Q$ is accordant, then every proper component of $kG/P$ is accordant.*

*Proof.* By Frobenius Reciprocity, every simple $kG$-module lies in $kG/Q$. By Lemma 1.1, every weight $kG$-module lies in $kG/S$. But $kG/S \dashv kG/Q$, so every weight $kG$-module lies in $kG/Q$. Since $kG/Q \dashv kG/P$, the required conclusion now follows from Remark 1.5. $\square$

Therefore, if Conjecture 1.2 holds when $P = S$, then it holds for all $p$-subgroups $P$ of $G$ and, in particular, the block form of Alperin's Conjecture holds for $kG$.

Let us point out a connection with Naehrig [Nae10]. When two indecomposable direct summands $U$ and $V$ of $kG/S$ are equivalent in the sense of [Nae10, 4.1(b)], the corresponding principal indecomposable modules of $\mathrm{End}_{kG}(kG/S)$ lie in the same block of $\mathrm{End}_{kG}(kG/S)$, hence $U$ and $V$ lie in the same connected component of $kG/S$.

In Section 2, we shall illustrate the conjecture with some examples. In Section 3, we shall deal with two special cases. We shall show that, when $G$ has a split BN-pair of characteristic $p$, the Cabanes–Sawada Theorem immediately implies that the conjecture holds for the Sylow permutation $kG$-module. We shall also show that, letting $T$ be a Sylow $p$-subgroup of the normalizer of a cyclic defect group of a block $b$ of $kG$, then the conjecture holds for the proper components of $bkG/T$.

The conjecture originates in [Bar91]. Though not mentioned in [Bar94], it was one of the motives for the defect theory, in [Bar94], for blocks of endomorphism algebras.

# 2 Some examples

In this section, to illustrate Conjecture 1.2, we present the structure of the Sylow permutation module in two particular cases.

First put $p = 2$ and $G = A_7$. Using the MAGMA source code in Zimmermann's thesis [Zim04], it can be shown that, over the field $\mathbb{F}_2$ of order 2, the 2-Sylow permutation module has the depicted stucture, where $n$ denotes an $n$-dimensional simple $\mathbb{F}_2G$-module and $n^*$ denotes its dual.

$$(1) \oplus (14) \oplus \begin{pmatrix} 14 & 20 \\ 1 & 1 \\ 14 & 20 \end{pmatrix} \oplus 2 \begin{pmatrix} 20 \\ 1 \\ 14 \\ 1 \\ 20 \end{pmatrix} \oplus \begin{pmatrix} & 14 & \\ & 1 & \\ 14 & & 20 \\ & 1 & \\ & 14 & \end{pmatrix}$$

$$\oplus (6) \oplus \begin{pmatrix} 4^* \\ 6 \\ 4 \end{pmatrix} \oplus \begin{pmatrix} 4 \\ 6 \\ 4^* \end{pmatrix} \oplus \begin{pmatrix} & 6 & \\ 4 & & 4^* \\ & 6 & \end{pmatrix}.$$

Using Zimmermann's MAGMA routines or, alternatively, using data in Benson [Ben84, Appendix], it can be shown that all 6 of the simple $\mathbb{F}_2G$-modules are absolutely simple.

Again using MAGMA or [Ben84, Appendix], it can be shown that the indecomposable summands with Loewy length 5 are projective and therefore cannot be weight modules. The non-simple indecomposable summand with socle 6 has vertex $V_4$ and has a 4-dimensional non-simple Green correspondent, so this summand is not a weight module. But the simple summand 6 and the indecomposable summands with socles 4 and $4^*$ all have vertex $V_4$ and Green correspondents that are 2-dimensional, absolutely simple and inflated from projective modules. So those three summands are weight modules. Similarly, the simple summands 1 and 14 and the indecomposable summand with socle $14 + 20$ are weight modules. Evidently, the proper components of the Sylow permutation module have dimensions 1, 260, 54 with 1,

2, 3 isomorphism classes of simple modules and 1, 2, 3 isomorphism classes of weight modules lying in them.

Let us give an example where the partitioning of simple modules and weight modules into blocks of $\mathrm{End}_{kG}(kG/S)$ is much finer than the partitioning into blocks of $kG$. Using MAGMA or [Ben84, Appendix], it is not hard to show that, for $p = 3$ and $G = M_{10}$, the 3-Sylow permutation module has the structure

$$(1) \oplus (1_-) \oplus \begin{pmatrix} & 4 & \\ 1 & & 1_- \\ & 4^* & \end{pmatrix} \oplus \begin{pmatrix} & 4^* & \\ 1 & & 1_- \\ & 4 & \end{pmatrix} \oplus 2 \begin{pmatrix} & 6 & \\ 4 & & 4^* \\ & 6 & \end{pmatrix} \oplus (9_1) \oplus (9_2)\,.$$

In this case, the principal block of $kG$ contains 4 of the proper components.

The authors have also confirmed that Conjecture 1.2 holds for the groups $S_6$, $A_7$, $L_2(25)$, $M_{11}$, $J_1$ in characteristic 2, for $S_6$, $S_7$, $A_8$, $L_3(4)$, $L_2(25)$, $M_{11}$ in characteristic 3, and for McL in characteristic 5. Using data in Lempken–Staszewski [LS93], it can be shown that, in the principal 5-block of McL, three of the weight modules have socles of the form $2.250 + 896_2$ and $2.560 + 3038 + 3245_1 + 3245_2$ and $896_1 + 3.3038$.

# 3 Proof in two special cases

Let us first show that the conjecture holds in the scenario of the Cabanes–Sawada Theorem.

**Theorem 3.1.** (Cabanes–Sawada) *Suppose that $G$ has a split BN-pair of characteristic $p$. Let $S$ be a Sylow $p$-subgroup of $G$. Then:*
**(1)** *Every indecomposable direct summand of $kG/S$ is a weight $kG$-module. Every weight $kG$-module occurs with multiplicity 1 in $kG/S$.*
**(2)** *There is a bijective correspondence between the isomorphism classes of simple $kG$-modules $U$ and the isomorphism classes of weight $kG$-modules $W$ such that the isomorphism classes of $U$ and $W$ correspond provided $U \cong \mathrm{soc}(W)$.*

*In particular, every proper component of $kG/S$ is accordant.*

*Proof.* This follows from Cabanes [Cab88, Proposition 8], which says that the weak form of Alperin's Conjecture holds for $kG$, and Sawada [Saw77, 2.8], which says that every simple $kG$-module has multiplicity 1 in $\mathrm{soc}(kG/S)$. □

For another approach towards simultaneously refining Alperin's Conjecture and generalizing the Cabanes-Sawada Theorem, see [Nae10, Section 3]. We now turn to the case of a block with a cyclic defect group.

**Theorem 3.2.** *Let $b$ be a block of $kG$ with a cyclic defect group $D$. Let $T$ be a Sylow $p$-subgroup of $N_G(D)$. Then every proper component of $bkG/T$ is accordant.*

*Proof.* Erdmann's Theorem [Erd77] asserts that, given a simple $kG$-module $V$ with cyclic vertex $Q$, then $Q$ is the defect group of the block of $kG$ containing $V$. Hence, using the compatibility of the Green correspondence and the Brauer correspondence, as recorded in Alperin [Alp86, 14.4], it is easy to show that every simple $kGb$-module and every weight $kGb$-module has vertex $D$.

We may assume that $D$ is non-trivial. Let $E$ be the smallest non-trivial subgroup of $D$. Suppose that $E \trianglelefteq G$. Given a subgroup $L$ of $G$ containing $E$, we write $\overline{L} = L/E$. Let $\overline{b}$ be

the image of $b$ under the canonical epimorphism $kG \to k\overline{G}$. The simple $kGb$-modules, all of which have vertex $D$, are the inflations of the simple $k\overline{G}$-modules, all of which have vertex $\overline{D}$. Writing $\overline{b} = \sum_i b_i$ as a sum of blocks $b_i$ of $k\overline{G}$, then all the blocks $b_i$ have defect group $\overline{D}$. Since $\overline{T}$ is a Sylow $p$-subgroup of the group $N_{\overline{G}}(\overline{D}) = \overline{N_G(D)}$, an inductive argument on $|D|$ allows us to assume that every proper component of $\overline{b}k\overline{G}/\overline{T}$ is accordant. Observing that $\overline{b}k\overline{G}/\overline{T}$ inflates to $bkG/T$, we deduce that $bkG/T$ is accordant in the case $E \trianglelefteq G$.

Now suppose that $E$ is not normal in $G$. Let $H = N_G(E)$. Since $D$ is cyclic, $N_G(D) \leq H$. Let $c$ be the block of $kH$ with defect group $P$ such that $c$ is in Brauer correspondence with $b$. By Erdmann's Theorem combined with the compatibility of the Green correspondence and the Brauer correspondence again, the Green correspondence, with respect to vertex $D$, restricts to a bijective correspondence between the isomorphism classes of weight $kHc$-modules and the isomorphism classes of weight $kGb$-modules. Green [Gre74, Theorem 1(ii)] says that the isomorphism classes of simple $kHc$-modules $V$ are in a bijective correspondence with the isomorphism classes of simple $kGb$-modules $U$ whereby $V \leftrightarrow U$ provided $U$ is isomorphic to the socle of the Green correspondent of $V$.

Let $W$ be a weight $kHc$-module and let $V$ be a simple $kHc$-module. Let $\mathcal{G}(W)$ and $\mathcal{G}(V)$ denote the $kGb$-modules in Green correspondence with $W$ and $V$, respectively. By the previous paragraph, $\mathcal{G}(W)$ is a weight $kGb$-module and $\mathcal{G}(V)$ is an indecomposable $kGb$-module with a unique simple submodule $V_G$. Supposing that $W$ and $V$ lie in the same proper component of the $kH$-module $kH/T$ then, by [Bar94, Corollary 5.7(b)], $\mathcal{G}(W)$ and $\mathcal{G}(V)$ lie in the same proper component of the $kG$-module $kG/T \cong {}_G\mathrm{Ind}_H(kH/T)$. Plainly, $\mathcal{G}(W)$ and $V_G$ lie in the same proper component of $kG/T$. We have shown that, given a weight $kHc$-module and a simple $kHc$-module lying in the same proper component of $kH/T$, then the corresponding weight $kGb$-module and simple $kGb$-module lie in the same proper component of $kG/T$. The required conclusion for $bkG/T$ now follows because, by an inductive argument on $|G|$, we may assume that the required conclusion holds for $ckH/T$. $\square$

# References


[Alp86] J. L. Alperin, "Local Representation Theory", (Cambridge Univ. Press, Cambridge, 1986).

[Alp87] J. L. Alperin, *Weights for finite groups*, Symp. Pure Math. **47**, 369-379 (1987).

[Bar91] L. Barker, *Blocks of endomorphism algebras of modules*, PhD Thesis, University of Oxford, Oxford (1991).

[Bar94] L. Barker, *Blocks of endomorphism algebras*, J. Alg. **168**, 728-740 (1994).

[Ben84] D. Benson, "Modular Representation Theory: New Trends and Methods", Springer Lecture Notes in Math. 1081, (Springer, Berlin, 1984).

[Cab88] M. Cabanes, *Brauer morphism between modular Hecke algebras*, J. Alg. **115**, 1-31 (1988).

[Erd77] K. Erdmann, *Blocks and simple modules with cyclic vertices*, Bull. Lond. Math. Soc. **9**, 216-218 (1977).

[Gre74] J. A. Green, *Walking around the Brauer tree*, J. Austral. Math. Soc. **17**, 197-213 (1974).

[LS93] W. Lempken, R. Staszewski, *Some 5-modular representation theory for the simple group McL*, Comm. Alg. **21**, 1611-1629 (1993).

[Nae10] N. Naehrig, *Endomorphism rings of permutation modules*, J. Alg. **324**, 1044-1075 (2010).

[Saw77] H. Sawada, *A characterization of the modular representations of a finite group with split BN-pairs*, Math. Zeit. **155**, 29-41 (1977).

[Zim04] R. Zimmermann, *Vertizes einfacher Moduln Symmetrischer Gruppen*, PhD Thesis, University of Jena, Jena (2004).