Real representation spheres and the real monomial Burnside ring

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Abstract

We introduce a restriction morphism, called the Boltje morphism, from a given ordinary representation functor to a given monomial Burnside functor. In the case of a sufficiently large fibre group, this is Robert Boltje's splitting of the linearization morphism. By considering a monomial Lefschetz invariant associated with real representation spheres, we show that, in the case of the real representation ring and the fibre group $\{\pm 1\}$, the image of a modulo 2 reduction of the Boltje morphism is contained in a group of units associated with the idempotents of the 2-local Burnside ring. We deduce a relation on the dimensions of the subgroup-fixed subspaces of a real representation.

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1 Introduction

We shall be making a study of some restriction morphisms which, at one extreme, express Boltje's canonical induction formula [Bol90] while, at the other extreme, they generalize a construction initiated by tom Dieck [Die79, 5.5.9], namely, the tom Dieck morphism associated with spheres of real representations. A connection between canonical induction and the tom Dieck morphism has appeared before, in Symonds [Sym91], where the integrality property of Boltje's restriction morphism was proved by using the natural fibration of complex projective space as a monomial analogue of the sphere.

Generally, our concern will be with finite-dimensional representations of a finite group G over a field $\mathbb{K}$ of characteristic zero. A little more specifically, our concern will be with the old idea of trying to synthesize information about $\mathbb{K}G$-modules from information about certain 1-dimensional $\mathbb{K}I$-modules where I runs over some or all of the subgroups of G. Throughout, we let C be a torsion subgroup of the unit group $\mathbb{K}^\times = \mathbb{K} - \{0\}$. The 1-dimensional $\mathbb{K}I$-modules to which we shall be paying especial attention will be those upon which each element of I acts as multiplication by an element of C. Some of the results below are specific to the case where $\mathbb{K} = \mathbb{R}$ and $C = \{\pm 1\}$, and some of them are also specific to the case where G is a 2-group.

Fixing C, we write $O_C(G)$, or just $O(G)$, to denote the smallest normal subgroup of G such that the quotient group $G/O(G)$ is abelian and every element of $G/O(G)$ has the same order as some element of C. In other words, $O(G)$ is intersection of the kernels of the group homomorphisms $G \to C$.

Consider a $\mathbb{K}G$-module M, finite-dimensional as we deem all $\mathbb{K}G$-modules to be. Given a subgroup $I \leq G$, then the $O(I)$-fixed subspace $M^{O(I)}$ of M is the sum of those 1-dimensional $\mathbb{K}I$-submodules of M that are inflated from $I/O(I)$. For elements $c \in C$ and $i \in I$, we write $M_c^{I,i}$ to denote the c-eigenspace of the action of i on $M^{O(I)}$. By Maschke's Theorem,

$$M^{O(I)} = \bigoplus_{c \in C} M_c^{I,i} \, , \qquad \dim(M^{O(I)}) = \sum_{c \in C} \dim(M_c^{I,i}) \, .$$

We shall introduce a restriction morphism, denoted $\dim^c$, whereby the isomorphism class $[M]$ of M is associated with the function

$$(I, i) \mapsto \dim(M_c^{I,i}) \, .$$

We shall define the *Boltje morphism* to be the restriction morphism

$$\mathrm{bol}^{\mathbb{K},C} = \sum_{c \in C} c \, \dim^c \, .$$

This morphism is usually considered only in the case where C is sufficiently large in the sense that each element of G has the same order as some element of C. In that case, the field $\mathbb{K}$ splits for G, the Boltje morphism is a splitting for linearization and we have a canonical induction formula. At the other extreme though, when $C = \{1\}$, the monomial dimension morphism $\dim^1$ is closely related to the tom Dieck morphism die(), both of those morphisms associating the isomorphism class $[M]$ with the function

$$I \mapsto \dim_{\mathbb{R}}(M^I) \, .$$

The vague comments that we have just made are intended merely to convey an impression of the constructions. In Section 2, we shall give details and, in particular, we shall be elucidating those two extremal cases.

For the rest of this introductory section, let us confine our discussion to the case where we have the most to say, the case $\mathbb{K} = \mathbb{R}$. Here, the only possibilities for C are $C = \{1\}$ and $C = \{\pm 1\}$. We shall be examining the modulo 2 reductions of the morphisms $\dim^c$ and $\mathrm{bol}^{\mathbb{R},C}$. We shall be making use of the following topological construction. Given an $\mathbb{R}G$-module M, we let $S(M)$ denote the unit sphere of M with respect to any G-invariant inner product on M. Up to homotopy, $S(M)$ can be regarded as the homotopy G-sphere obtained from M by removing the zero vector.

Let us make some brief comments concerning the case $C = \{1\}$. The reduced tom Dieck morphism $\overline{\mathrm{die}}$ is so-called because it can be regarded as a modulo 2 reduction of the tom Dieck morphism die(). Via $\overline{\mathrm{die}}$, the isomorphism class $[M]$ is associated with the function

$$I \mapsto \mathrm{par}(\dim(M^I))$$

where $\mathrm{par}(n) = (-1)^n$ for $n \in \mathbb{Z}$. We can view $\overline{\mathrm{die}}$ as a morphism of biset functors

$$\overline{\mathrm{die}} \, : \, A_{\mathbb{R}} \to \beta^{\times}$$

where the coordinate module $A_{\mathbb{R}}(G)$ is the real representation ring of G and the coordinate module $\beta^{\times}(G)$ is the unit group of the ghost ring $\beta(G)$ associated with the Burnside ring $B(G)$ of G. But we shall be changing the codomain. A result of tom Dieck asserts that the image of the coordinate map $\overline{\mathrm{die}}_G : A_{\mathbb{R}}(G) \to \beta^{\times}(G)$ is contained in the unit group $B^{\times}(G)$ of $B(G)$. His proof makes use of the fact that the function $I \mapsto \mathrm{par}(\dim(M^I))$ is determined by the Lefchetz invariant of $S(M)$. Hence, we can regard the reduced tom Dieck morphism as a morphism of biset functors

$$\overline{\mathrm{die}} : A_{\mathbb{R}} \to B^{\times} .$$

The main substance of this paper concerns the case $C = \{\pm 1\}$, still with $\mathbb{K} = \mathbb{R}$. We now replace the ordinary Burnside ring $B(G)$ with the *real Burnside ring* $B_{\mathbb{R}}(G) = B_{\{\pm 1\}}(G)$, we mean to say, the monomial Burnside ring with fibre group $\{\pm 1\}$. For the rest of this section, we assume that $C = \{\pm 1\}$. Thus, the group $O(G) = O_C(G)$ is the smallest normal subgroup of G such that $G/O(G)$ is an elementary abelian 2-group. We write $O^2(G)$ to denote the smallest normal subgroup of G such that $G/O^2(G)$ is a 2-group.

In a moment, we shall define a restriction morphism $\overline{\mathrm{bol}}$, called the **reduced Boltje morphism**, whereby $[M]$ is associated with the function

$$I \mapsto \mathrm{par}(\dim(M^{O(I)})) .$$

Some more notation is needed. Recall that the algebra maps $\mathbb{Q}B(G) \to \mathbb{Q}$ are the maps $\epsilon_I^G : \mathbb{Q}B(G) \to \mathbb{Q}$, indexed by representatives I of the conjugacy classes of subgroups of G, where $\epsilon_I^G[\Omega] = |\Omega^I|$, the notation indicating that the isomorphism class $[\Omega]$ of a G-set Ω is sent to the number of I-fixed elements of Ω. Also recall that any element x of $\mathbb{Q}B(G)$ has coordinate decomposition

$$x = \sum_I \epsilon_I^G(x)\, e_I^G$$

where each e_I^G is the unique primitive idempotent of $\mathbb{Q}B(G)$ such that $\epsilon_I^G(e_I^G) \neq 0$. The ghost ring $\beta(G)$ is defined to be the set consisting of those elements x such that each $\epsilon_I^G(x) \in \mathbb{Z}$. Evidently, the unit group $\beta^{\times}(G)$ of $\beta(G)$ consists of those elements x such that each $\epsilon_I^G(x) \in \{\pm 1\}$. In particular, $\beta^{\times}(G)$ is an elementary abelian 2-group, and it can be regarded as a vector space over the field of order 2. Our notation follows [Bar10, Section 3], where fuller details of these well-known constructions are given. We define $\overline{\mathrm{bol}}_G : A_{\mathbb{R}}(G) \to \beta^{\times}(G)$ to be the $\mathbb{Q}$-linear map such that

$$\overline{\mathrm{bol}}_G[M] = \sum_I \mathrm{par}(\dim(M^{O(I)}))\, e_I^G .$$

Evidently, we can view $\overline{\mathrm{bol}}$ as a morphism of restriction functors $A_{\mathbb{R}} \to \beta^{\times}$. Extending to the ring $\mathbb{Z}_{(2)}$ of 2-local integers, we can view $\overline{\mathrm{bol}}$ as a morphism of restriction functors $\mathbb{Z}_{(2)}A_{\mathbb{R}} \to \beta^{\times}$.

Let $\beta^{\times}_{(2)}$ denote the restriction subfunctor of $\beta^{\times}$ such that $\beta^{\times}_{(2)}(G)$ consists of those units in $\beta^{\times}(G)$ which can be written in the form $1 - 2y$, where y is an idempotent of $\mathbb{Z}_{(2)}B(G)$. In analogy with the above result of tom Dieck, we shall prove the following result in Section 3.

Theorem 1.1. *The image of the map $\overline{\mathrm{bol}}_G : \mathbb{Z}_{(2)}A_{\mathbb{R}}(G) \to \beta^{\times}(G)$ is contained in $\beta^{\times}_{(2)}(G)$. Hence, $\overline{\mathrm{bol}}$ can be regarded as a restriction morphism $\overline{\mathrm{bol}} : \mathbb{Z}_{(2)}A_{\mathbb{R}} \to \beta^{\times}_{(2)}$.*

In Section 4, using Theorem 1.1 together with a characterization of idempotents due to Dress, we shall obtain the following result. We write $\equiv_2$ to denote congruence modulo 2.

Theorem 1.2. *Given an $\mathbb{R}G$-module M, then $\dim(M^{O(I)}) \equiv_2 \dim(M^{O^2(I)})$ for all $I \leq G$.*

Specializing to the case of a finite 2-group, and using a theorem of Tornehave, we shall deduce the next result, which expresses a constraint on the units of the Burnside ring of a finite 2-group. We shall also give a more direct alternative proof, using the same theorem of Tornehave and also using an extension in [Bar06] of Bouc's theory [Bou10, Chapter 9] of genetic sections.

Theorem 1.3. *Suppose that G is a 2-group. Then, for all $I \leq G$ and all units $x \in B^\times(G)$, we have $\epsilon^G_{O(I)}(x) = \epsilon^G_1(x)$.*

2 Boltje morphisms

For an arbitrary field $\mathbb{K}$ with characteristic zero, an arbitrary torsion subgroup C of the unit group $\mathbb{K}^\times$ and an arbitrary element $c \in C$, we shall define a restriction morphism $\dim^c$, called the *monomial dimension morphism* for eigenvalue c, and we shall define a restriction morphism $\mathrm{bol}^{C,\mathbb{K}}$, called the *Boltje morphism* for C and $\mathbb{K}$. In this section, we shall explain how, in one extremal case, $\mathrm{bol}^{C,\mathbb{K}}$ is associated with canonical induction while, in another extremal case, $\mathrm{bol}^{C,\mathbb{K}}$ is associated with dimension functions on real representation spheres.

We shall be considering three kinds of group functors, namely, restriction functors, Mackey functors, biset functors. All of our group functors are understood to be defined on the class of all finite groups, except when we confine attention to the class of all finite 2-groups. For any group functor L, we write $L(G)$ for the coordinate module at G. For any morphism of group functors $\theta : L \to L'$, we write $\theta_G : L(G) \to L'(G)$ for the coordinate map at G. Any group isomorphism $G \to G'$, gives rise to an isogation map (sometimes awkwardly called an isomorphism map) $L(G) \to L(G')$, which is to be interpreted as transport of structure. Restriction functors are equipped with isogation maps and restriction maps. Mackey functors are further equipped with induction maps, biset functors are yet further equipped with inflation and deflation maps. A good starting-point for a study of these briefly indicated notions is Bouc [Bou10].

Recall that the representation ring of the group algebra $\mathbb{K}G$ coincides with the character ring of $\mathbb{K}G$. Denoted $A_\mathbb{K}(G)$, it is a free $\mathbb{Z}$-module with basis $\mathrm{Irr}(\mathbb{K}G)$, the set of isomorphism classes of simple $\mathbb{K}G$-modules, which we identify with the set of irreducible $\mathbb{K}G$-characters. The sum and product on $A_\mathbb{K}(G)$ are given by direct sum and tensor product. We can understand $A_\mathbb{K}$ to be a biset functor for the class of all finite groups, equipped with isogation, restriction, induction, inflation, deflation maps. Actually, the inflation and deflation maps will be of no concern to us in this paper, and we can just as well regard $A_\mathbb{K}(G)$ as a Mackey functor, equipped only with isogation, restriction and induction maps.

The monomial Burnside ring of G with fibre group C, denoted $B_C(G)$, is defined similarly, but with C-fibred G-sets in place of $\mathbb{K}G$-modules. Recall that a **C-fibred G-set** is a permutation set Ω for the group $CG = C \times G$ such that C acts freely and the number of C-orbits is finite. A C-orbit of Ω is called a **fibre** of Ω. It is well-known that B_C can be regarded as a biset functor. For our purposes, we can just as well regard it as a Mackey functor.

Let us briefly indicate two coordinate decompositions that were reviewed in more detail in [Bar04, Equations 1, 2]. Defining a **C-subcharacter** of G to be a pair (U, μ) where $U \leq G$ and $\mu \in \mathrm{Hom}(U, C)$, then we have a coordinate decomposition

$$B_C(G) = \bigoplus_{(U,\mu)} \mathbb{Z}\, d^G_{U,\mu}$$

where (U,μ) runs over representatives of the G-conjugacy classes of C-subcharacters, and $d^G_{U,\mu}$ is the isomorphism class of a transitive C-fibred G-set such that U is the stabilizer of a fibre and U acts via μ on that fibre. The other coordinate decomposition concerns the algebra $\mathbb{K}B_C(G) = \mathbb{K} \otimes B_C(G)$. We define a C-**subelement** of G to be a pair $(I, iO_C(I))$, where $i \in I \leq G$. As an abuse of notation, we write (I,i) instead of $(I, iO_C(I))$. For each C-subelement (I,i), let $\epsilon^G_{I,i}$ be the algebra map $\mathbb{K}B_C(G) \to \mathbb{K}$ associated with (I,i). Recall that, given a C-fibred G-set Ω, then $\epsilon^G_{I,i}[\Omega] = \sum_\omega \phi_\omega$, where ω runs over the fibres stabilized by I and i acts on ω as multiplication by ϕ_ω. Let $e^G_{I,i}$ be the unique primitive idempotent of $\mathbb{K}B_C(G)$ such that $\epsilon^G_{I,i}(e^G_{I,i}) = 1$. Note that we have G-conjugacy $(I,i) =_G (J,j)$ if and only if $\epsilon^G_{I,i} = \epsilon^G_{J,j}$, which is equivalent to the condition $e^G_{I,i} = e^G_{J,j}$. We have

$$\mathbb{K}B_C(G) = \bigoplus_{(I,i)} \mathbb{K}\, e^G_{I,i}$$

where (I,i) runs over representatives of the G-conjugacy classes of C-subelements. Thus, given an element $x \in \mathbb{K}B_C(G)$, then

$$x = \sum_{(I,i)} \epsilon^G_{I,i}(x)\, e^G_{I,i}\ .$$

Recall that there is an embedding $B(G) \hookrightarrow B_C(G)$ such that $[\mho] \mapsto [C\mho]$, where each element ω of a given G-set $\mho$ corresponds to a fibre $\{c\omega : c \in C\}$ of the C-fibred G-set $C\mho = C \times \mho$. The embedding is characterized by an easy remark [Bar04, 7.2], which says that, given $x \in B_C(G)$, then $x \in B(G)$ if and only if $\epsilon^G_{I,i}(x) = \epsilon^G_{I,i'}(x)$ for all $i, i' \in I$, in which case, $\epsilon^G_I(x) = \epsilon^G_{I,i}(x)$ for all $i \in I$. We shall be needing the following remark in the next section.

Remark 2.1. *Let R be a unital subring of $\mathbb{K}$. Then $\mathbb{K}B(G) \cap RB_C(G) = RB(G)$.*

Proof. Let $\pi_C : B_C(G) \to B(G)$ be the projection such that $[\Omega] \mapsto [C\backslash\Omega]$, where $C\backslash\Omega$ denotes the G-set of fibres of a given C-fibred G-set Ω. Extending linearly, we obtain projections $\pi_C : RB_C(G) \to RB(G)$ and $\pi_C : \mathbb{K}B_C(G) \to \mathbb{K}B(G)$. Given $x \in \mathbb{K}B(G) \cap RB_C(G)$, then $x = \pi_C(x) \in RB(G)$. So $\mathbb{K}B(G) \cap RB_C(G) \subseteq RB(G)$. The reverse inclusion is obvious. $\square$

We mention that the projection $\pi_C : \mathbb{K}B_C(G) \to \mathbb{K}B(G)$ is an algebra map and, since $\epsilon^G_I[C\backslash\Omega] = \epsilon^G_{I,1}[\Omega]$, we have $\pi_C(e^G_{I,i}) = e^G_I$ if $i \in O(I)$ while $\pi_C(e^G_{I,i}) = 0$ otherwise.

We shall also be making use of the primitive idempotents of $\mathbb{K}A_\mathbb{K}(G)$. Regarding $\mathbb{K}A_\mathbb{K}(G)$ as the $\mathbb{K}$-vector space of G-invariant functions $G \to \mathbb{K}$, then the algebra maps $\mathbb{K}A_\mathbb{K}(G) \to \mathbb{K}$ are the maps ϵ^G_g, indexed by representatives g of the conjugacy classes of G, where $\epsilon^G_g(\chi) = \chi(g)$ for $\chi \in \mathbb{K}A_\mathbb{K}(G)$. Letting e^G_g be the primitive idempotent such that $\epsilon^G_g(e^G_g) = 1$, then

$$\chi = \sum_g \epsilon^G_g(\chi)\, e^G_g = \sum_g \chi(g)\, e^G_g$$

where g runs over representatives of the conjugacy classes of G. The linearization morphism

$$\mathrm{lin}^{C,\mathbb{K}}\ :\ \mathbb{K}B_C \to \mathbb{K}A_\mathbb{K}$$

has coordinate morphisms $\mathrm{lin}^{C,\mathbb{K}}_G : \mathbb{K}B_C(G) \to \mathbb{K}A_\mathbb{K}(G)$ such that

$$\mathrm{lin}^{C,\mathbb{K}}_G[d^G_{U,\mu}] = \mathrm{ind}_{G,U}(\mu)\,.$$

Letting Ω be a C-fibred G-set, and letting $\mathbb{K}\Omega = \mathbb{K} \otimes_C \Omega$ be the evident extension of Ω to a $\mathbb{K}G$-module, then $\mathrm{lin}^{C,\mathbb{K}}_G[\Omega] = [\mathbb{K}\Omega]$.

Remark 2.2. *Given a primitive idempotent $e^G_{I,i}$ of $\mathbb{K}B_C(G)$, then $\mathrm{lin}^{C,\mathbb{K}}_G(e^G_{I,i}) \neq 0$ if and only if I is cyclic with generator i, in which case $\mathrm{lin}^{C,\mathbb{K}}_G(e^G_{I,i}) = e^G_i$.*

Proof. It suffices to show that $\epsilon^G_{\langle i \rangle, i}[\Omega] = \epsilon^G_i[\mathbb{K}\Omega]$. Letting x run over representatives of the fibres of Ω, then x runs over the elements of a basis for the $\mathbb{K}G$-module $\mathbb{K}\Omega$. With respect to that basis, the action of i on $\mathbb{K}\Omega$ is represented by a matrix which has exactly one entry in each row and likewise for each column. The two sides of the required equation are plainly both equal to the trace of that matrix. □

Given $c \in C$, we define a $\mathbb{K}$-linear map

$$\dim^c_G \; : \; \mathbb{K}A_{\mathbb{K}}(G) \to \mathbb{K}B_C(G)$$

such that $\epsilon^G_{I,i}(\dim^c_G[M]) = \dim(M^{I,i}_c)$ for a $\mathbb{K}G$-module M. In other words,

$$\dim^c_G[M] = \sum_{(I,i)} \dim(M^{I,i}_c)\, e^G_{I,i} \, .$$

Since $\epsilon^H_{I,i}(\mathrm{res}_{H,G}(x)) = \epsilon^G_{I,i}(x)$ for all intermediate subgroups $I \leq H \leq G$, the maps $\dim^c_G$ commute with restriction. Plainly, the maps $\dim^c_G$ also commute with isogation. Thus, the maps $\dim^c_G$ combine to form a restriction morphism

$$\dim^c \; : \; \mathbb{K}A_{\mathbb{K}} \to \mathbb{K}B_C \, .$$

Let us define the **Boltje morphism** to be the restriction morphism

$$\mathrm{bol}^{C,\mathbb{K}} = \sum_{c \in C} c \dim^c \; : \; \mathbb{K}A_{\mathbb{K}} \to \mathbb{K}B_C \, .$$

The sum makes sense because, for each G, the sum $\mathrm{bol}^{C,\mathbb{K}}_G = \sum_{c \in C} c \dim^c_G$ is finite, indeed, $\dim^c_G$ vanishes for all c whose order does not divide $|G|$. When C is sufficiently large, the Boltje morphism is a splitting for linearization. We mean to say, if every element of G has the same order as an element of C, then

$$\mathrm{lin}^{C,\mathbb{K}}_G \circ \mathrm{bol}^{C,\mathbb{K}}_G = \mathrm{id}_{\mathbb{K}A_{\mathbb{K}}(G)} \, .$$

To see this, first note that, for arbitrary C and $\mathbb{K}$, we have

$$\mathrm{bol}^{C,\mathbb{K}}_G[M] = \sum_{(I,i)} \chi_I(i) e^G_{I,i}$$

where χ_I is the $\mathbb{K}I$-character of the $\mathbb{K}I$-module $M^{O(I)}$. Using Remark 2.2,

$$\mathrm{lin}^{C,\mathbb{K}}_G(\mathrm{bol}^{C,\mathbb{K}}_G[M]) = \sum_{(I,i)} \chi_I(i) \, \mathrm{lin}^{C,\mathbb{K}}_G(e_{I,i}) = \sum_i \chi(i) \, e_i$$

where χ is the $\mathbb{K}G$-character of M and, in the final sum, i runs over representatives of those conjugacy classes of elements of G such that the order of i divides $|G|$. When C is sufficiently large in the sense specified above, i runs over representatives of all the conjugacy classes, and $\sum_i \chi(i) \, e_i = [M]$, as required.

Let us confirm that the assertion we have just made is just a reformulation of the splitting result in Boltje [Bol90]. Suppose, again, that C is sufficiently large. Then, in particular, $\mathbb{K}$ is a splitting field for G. We must now resolve two different notations. Where we write $B_C(G)$ and $A_{\mathbb{K}}(G)$ and $\mathrm{lin}_G^{C,\mathbb{K}}$ and $d_{U,\mu}^G$, Boltje [Bol90] writes $R_+(G)$ and $R(G)$ and b_G and $\overline{(U,\mu)}^G$, respectively. Note that, because of the hypothesis on C, the scenario is essentially independent of C and $\mathbb{K}$. In [Bol90, 2.1], he shows that there exists a unique restriction morphism $a : A_{\mathbb{K}} \to B_C$ such that $a_G(\phi) = d_{G,\phi}^G$ for all $\phi \in \mathrm{Hom}(G, C)$. Since

$$\epsilon_{I,i}^G(\mathrm{bol}_G^{C,\mathbb{K}}(\phi)) = \phi(i) = \epsilon_{I,i}^G(d_{G,\phi}^G) = \epsilon_{I,i}^G(a_G(\phi))$$

we have $\mathrm{bol}_G^{C,\mathbb{K}} = a_G$ and $\mathrm{bol}^{C,\mathbb{K}} = a$. But the splitting property that we have been discussing is just a preliminary to a deeper result about integrality. Having resolved the two different notations, we can now interpret Boltje [Bol90, 2.1(b)] as the following theorem, which expresses the integrality property too.

Theorem 2.3. (Boltje) *Suppose that every element of G has the same order as an element of C. Then the restriction morphism $\mathrm{bol}^{C,\mathbb{K}} : \mathbb{K}A_{\mathbb{K}} \to \mathbb{K}B_C$ is the $\mathbb{K}$-linear extension of the unique restriction morphism $\mathrm{bol}^{C,\mathbb{K}} : A_{\mathbb{K}} \to B_C$ such that $\mathrm{lin}^{C,\mathbb{K}} \circ \mathrm{bol}^{C,\mathbb{K}} = \mathrm{id}$.*

When the hypothesis on C is relaxed, the splitting property and the integrality property in the conclusion of the theorem can fail. Nevertheless, as we shall see in the next section, the Boltje morphism $\mathrm{bol}^{C,\mathbb{K}}$ does appear to be of interest even in the two smallest cases, where $C = \{1\}$ or $C = \{\pm 1\}$. Let us comment on a connection between the tom Dieck morphism die() and the Boltje morphism in the case $C = \{1\}$. Our notation die() is taken from a presentation in [Bar10, 4.1] of a result of Bouc–Yalçın [BY07, page 828]. Letting B^* denote the dual of the Burnside functor B, then the tom Dieck morphism $\mathrm{die} : A_{\mathbb{K}} \to B^*$ is given by

$$\mathrm{die}_G[M] = \sum_I \dim(M^I)\, \delta_I^G$$

where I runs over representatives of the G-conjugacy classes of subgroups of G, and the elements δ_I^G comprise a $\mathbb{Z}$-basis for $B^*(G)$ that is dual to the $\mathbb{Z}$-basis of $B(G)$ consisting of the isomorphism classes of transitive G-sets $d_I^G = [G/I]$. On the other hand, the morphism $\mathrm{bol}^{\{1\},\mathbb{K}} = \dim^1 : A_{\mathbb{K}} \to B$ is given by

$$\mathrm{bol}_G^{\{1\},\mathbb{K}}[M] = \dim_G^1[M] = \sum_I \dim(M^I)\, e_I^G\,.$$

Thus, although die() and $\mathrm{bol}^{\{1\},\mathbb{K}}$ have different codomains, their defining formulas are similar. A closer connection will transpire, however, when we pass to the reduced versions of those two morphisms in the special case $\mathbb{K} = \mathbb{R}$.

3 The reduced Boltje morphism

Still allowing the finite group G to be arbitrary, we now confine our attention to the case $\mathbb{K} = \mathbb{R}$. The only torsion units of $\mathbb{R}$ are 1 and -1, so the only possibilities for C are $C = \{1\}$ and $C = \{\pm 1\}$. We shall be discussing modulo 2 reductions of the tom Dieck morphism die() and the Boltje morphisms $\mathrm{bol}^{\{1\},\mathbb{R}}$ and $\mathrm{bol}^{\{\pm 1\},\mathbb{R}}$, realizing the reductions as morphisms by understanding their images to be contained in the unit groups $B^\times(G)$ and $\beta^\times(G)$, respectively. Although those unit groups are abelian, it will be convenient to write their group operations multiplicatively.

In preparation for a study of the case $C = \{\pm 1\}$, we first review the case $C = \{1\}$, drawing material from [Bar10] and Bouc–Yalçın [BY07]. The parity function $\mathrm{par} : n \mapsto (-1)^n$ is, of course, modulo 2 reduction of rational integers written multiplicatively (with the codomain C_2, the cyclic group with order 2, taken to be $\{\pm 1\}$ instead of $\mathbb{Z}/2\mathbb{Z}$). Thus, fixing an $\mathbb{R}G$-module M, and letting I run over representatives of the conjugacy classes of subgroups of G, the function $\overline{\mathrm{die}} : I \mapsto \mathrm{par}(\dim(M^I))$ is the modulo 2 reduction of the function $\mathrm{die} : I \mapsto \dim(M^I)$. In Section 2, we realized die() as a morphism with codomain B^*. But we shall be realizing $\overline{\mathrm{die}}$ as a morphism with codomain $B^\times$. Let us explain the relationship between those two codomains. Recall that the **ghost ring** associated with $B(G)$ is defined to be the $\mathbb{Z}$-span of the primitive idempotents $\beta(G) = \bigoplus_I \mathbb{Z}\, e_I^G$. We have $B(G) \leq \beta(G) < \mathbb{Q}B(G)$, and an element $x \in \mathbb{Q}B(G)$ belongs to $\beta(G)$ if and only if $\epsilon_I^G(x) \in \mathbb{Z}$ for each $I \leq G$. We also have an inclusion of unit groups $B^\times(G) \leq \beta^\times(G)$, and $x \in \beta^\times(G)$ if and only if each $\epsilon_I^G(x) \in \{\pm 1\}$. We shall be making use of Yoshida's characterization [Yos90, 6.5] of $B^\times(G)$ as a subgroup of $\beta^\times(G)$.

Theorem 3.1. (Yoshida's Criterion) *Given an element $x \in \beta^\times(G)$, then $x \in B^\times(G)$ if and only if, for all $I \leq G$, the function $N_G(I)/I \ni gI \mapsto \epsilon_{\langle I,g\rangle}^G(x)/\epsilon_I^G(x) \in \{\pm 1\}$ is a group homomorphism.*

As discussed in [Bar10, Section 10], the modulo 2 reduction of the biset functor B^* can be identified with the biset functor $\beta^\times$, and the modulo 2 reduction of the morphism of biset functors die() from $A_\mathbb{R}$ to B^* can be identified with the morphism of biset functors $\overline{\mathrm{die}}$ from $A_\mathbb{R}$ to $\beta^\times$ given by

$$\overline{\mathrm{die}}_G[M] = \sum_I \mathrm{par}(\dim(M^I))\, e_I^G \,.$$

A well-known result of tom Dieck asserts that the image $\overline{\mathrm{die}}_G(A_\mathbb{R}(G))$ is contained in $B^\times(G)$. Since $B^\times$ is a biset subfunctor of $\beta^\times$, we can regard $\overline{\mathrm{die}}$ as a morphism of biset functors

$$\overline{\mathrm{die}} \;:\; A_\mathbb{R} \to B^\times \,.$$

We call $\overline{\mathrm{die}}$ the **reduced tom Dieck morphism**. (In [Bar10], the tom Dieck morphism die() was called the "lifted tom Dieck morphism" for the sake of clear contradistinction.)

Below, our strategy for proving Theorem 1.1 will be to treat it as a monomial analogue of tom Dieck's inclusion $\overline{\mathrm{die}}(A_\mathbb{R}) \leq B^\times$. Just as an interesting aside, let us show how Yoshida's Criterion yields a quick direct proof of tom Dieck's inclusion. Consider an $\mathbb{R}G$-module M and an element $g \in G$. Let $m_+(g)$ and $m_-(g)$ be the multiplicities of 1 and -1, respectively, as eigenvalues of the action of g on M. Let $m(g)$ be the sum of the multiplicities of the non-real eigenvalues. Then $\dim(M) = m_+(g) + m_-(g) + m(g)$. Since the non-real eigenvalues occur in complex conjugate pairs, $m(g)$ is even and the determinant of the action of g is

$$\det(g : M) = \mathrm{par}(m_-(g)) = \mathrm{par}(m_+(g) - \dim(M)) = \frac{\mathrm{par}(\dim(M^{\langle g\rangle}))}{\mathrm{par}(\dim(M))} \,.$$

Let $x = \overline{\mathrm{die}}_G[M]$. Consider a subgroup $I \leq G$ and an element $gI \in N_G(I)/I$. Replacing the $\mathbb{R}G$-module M with the $\mathbb{R}N_G(I)/I$-module M^I, we have

$$\det(gI : M^I) = \frac{\mathrm{par}(\dim(M^{\langle I,g\rangle}))}{\mathrm{par}(\dim(M^I))} = \frac{\epsilon_{\langle I,g\rangle}^G(x)}{\epsilon_I^G(x)} \,.$$

By the multiplicative property of determinants, x satisfies the criterion in Theorem 3.1, hence $x \in B^\times(G)$. The direct proof of the inclusion $\overline{\text{die}}(A_\mathbb{R}) \leq B^\times$ is complete.

However, lacking an analogue of Theorem 3.1 for the case $C = \{\pm 1\}$, we shall be unable to adapt the argument that we have just given. Tom Dieck's original proof of the inclusion $\overline{\text{die}}(A_\mathbb{R}) \leq B^\times$ is well-known, but let us briefly present it. Let K be an admissible G-equivariant triangulation of the G-sphere $S(M)$. Thus, K is a G-simplicial complex, admissible in the sense that the stabilizer of any simplex fixes the simplex, and the geometric realization of K is G-homeomorphic to $S(M)$. Recall that the Lefschetz invariant of $S(M)$ is

$$\Lambda_G(S(M)) = \sum_{\sigma \in_G K} \text{par}(\ell(\sigma))\, [\text{Orb}_G(\sigma)]$$

as an element of $B(G)$, summed over representatives σ of the G-orbits of simplexes in K, where $\text{Orb}_G(\sigma)$ denotes the G-orbit of σ as a transitive G-set and $\ell(\sigma)$ denotes the dimension of σ. Here, we are not including any (-1)-simplex. For $I \leq G$, the subcomplex K^I consisting of the I-fixed simplexes is a triangulation of the I-fixed sphere $S(M)^I = S(M^I)$. Summing over all the simplexes σ in K^I, we have

$$\epsilon_I^G(\Lambda_G(S(M))) = \sum_{\sigma \in K^I} \text{par}(\ell(\sigma)) = \chi(S(M)^I) = 1 - \text{par}(\dim(M^I)) = \epsilon_I^G(1 - \overline{\text{die}}_G[M])$$

where χ denotes the Euler characteristic, equal to 2 or 0 for even-dimensional or odd-dimensional spheres, respectively. Therefore $\overline{\text{die}}_G[M] = 1 - \Lambda_G(S(M))$ and, perforce, $\overline{\text{die}}_G[M] \in B(G)$. But $\overline{\text{die}}_G[M] \in \beta^\times(G)$, hence $\overline{\text{die}}_G[M] \in B^\times(G)$. We have again established the inclusion $\overline{\text{die}}(A_\mathbb{R}) \leq B^\times$.

For the rest of this section, we put $C = \{\pm 1\}$. Thus, given a subgroup $I \leq G$, then $I/O(I)$ is the largest quotient group of I such that $I/O(I)$ is an elementary abelian 2-group. We shall prove Theorem 1.1 by adapting the above topological proof of the inclusion $\overline{\text{die}}(A_\mathbb{R}) \leq B^\times$.

Let M be an $\mathbb{R}G$-module. Allowing C to act multiplicatively on M and on $S(M)$, let K be an admissible CG-equivariant triangulation of $S(M)$. Thus, the hypothesis on K is stronger than before, the extra condition being that, when we identify the vertices of K with their corresponding points of $S(M)$, the vertices occur in pairs, z and $-z$. More generally, identifying the simplexes of K with their corresponding subsets of $S(M)$, the simplexes occur in pairs, σ and $-\sigma$, the points of any simplex being the negations of the points of its paired partner. As an element of $B_C(G)$, we define the **C-monomial Lefschetz invariant** of M to be

$$\Lambda_{CG}(M) = \sum_{\sigma} \text{par}(\ell(\sigma))\, [\text{Orb}_{CG}(\sigma)]$$

where σ now runs over representatives of the CG-orbits of simplexes in K, and $[\text{Orb}_{CG}(\sigma)]$ denotes the isomorphism class of the CG-orbit $\text{Orb}_{CG}(\sigma)$ as a C-fibred G-set. A similar monomial Lefchetz invariant, in the context of a sufficiently large fibre group, was considered by Symonds in [Sym91, Section 2]. To see that $\Lambda_{CG}(M)$ is an invariant of the CG-homotopy class of $S(M)$, observe that, regarding M as a CG-module and regarding $S(M)$ as a CG-space, then $\Lambda_{CG}(M)$ is determined by the usual Lefschetz invariant $\Lambda_{CG}(S(M)) \in B(CG)$, which is given by the same formula, but with $[\text{Orb}_{CG}(\sigma)]$ reinterpreted as the isomorphism class of $\text{Orb}_{CG}(\sigma)$ as a transitive CG-set.

Theorem 3.2. *Still assuming that $C = \{\pm 1\}$ and that M is an $\mathbb{R}G$-module then, for any*

C-subelement (I,i) of G, we have

$$\epsilon^G_{I,i}(\Lambda_{CG}(M)) = \sum_{\psi \in \mathrm{Irr}_M(\mathbb{R}I)} \psi(i)$$

where $\mathrm{Irr}_M(\mathbb{R}I)$ denotes the subset of $\mathrm{Irr}(\mathbb{R}I)$ consisting of those irreducible $\mathbb{R}I$-characters that have odd multiplicity in the $\mathbb{R}I$-module $M^{O(I)}$. In particular, $\epsilon^G_{I,i}(\Lambda_{CG}(M)) \equiv_2 \dim_{\mathbb{R}}(M^{O(I)})$.

Proof. We have $\dim_{\mathbb{R}}(M^{O(I)}) = \sum_\psi m_\psi$ where, for the moment, ψ runs over all the irreducible $\mathbb{R}I$-characters and m_ψ is the multiplicity of ψ in the $\mathbb{R}I$-character of $M^{O(I)}$. If $m_\psi \neq 0$ then ψ is the inflation of an irreducible $\mathbb{R}I/O(I)$-character and, in particular, $\psi(i) = \pm 1$. Therefore, $\dim_{\mathbb{R}}(M^{O(I)}) \equiv_2 \sum_\psi \psi(i)$, where ψ now runs over those irreducible $\mathbb{R}\overline{I}$-characters such that m_ψ is odd. So the rider will follow from the main equality.

Put $\Lambda = \Lambda_{CG}(M)$. Since $\epsilon^G_{I,i}(\Lambda) = \epsilon^I_{I,i}(\mathrm{res}_{I,G}(\Lambda)) = \epsilon^I_{I,i}(\Lambda_{CI}(\mathrm{res}_{I,G}(M)))$, we can replace M with $\mathrm{res}_{I,G}(M)$. In other words, we may assume that $I = G$. Let K be an admissible CG-equivariant triangulation of the sphere $S(M)$. We have

$$\epsilon^G_{G,i}(\Lambda) = \sum_\sigma \mathrm{par}(\ell(\sigma))\, \epsilon^G_{G,i}[\mathrm{Orb}_{CG}(\sigma)]$$

where σ runs over representatives of the CG-orbits of simplexes of K. By the definition of $\epsilon^G_{G,i}$, contributions to the sum come from only those representatives σ such that the fibre $\{\sigma, -\sigma\}$ is stabilized by G, in other words, $\{\sigma, -\sigma\} = \mathrm{Orb}_{CG}(\sigma)$. Let A be the set of simplexes ρ of K whose fibre is stablized by G. Let $\overline{G} = G/O(G)$, and regard the irreducible $\mathbb{R}\overline{G}$-characters as irreducible $\mathbb{R}G$-characters by inflation. For all $\rho \in A$, we have

$$\epsilon^G_{G,i}[\mathrm{Orb}_{CG}(\rho)] = \epsilon^G_{G,i}[\{\rho, -\rho\}] = \psi_\rho(i)$$

where ψ_ρ is the irreducible $\mathbb{R}\overline{G}$-character such that $i\rho = \psi_\rho(i)\rho$. Since each CG-orbit in A owns exactly two simplexes,

$$2\epsilon^G_{G,i}(\Lambda) = \sum_{\rho \in A} \psi_\rho(i)\, \mathrm{par}(\ell(\rho)) \ .$$

Defining $A_\psi = \{\rho \in A : \psi_\rho = \psi\}$, we have a disjoint union $A = \bigcup_\psi A_\psi$ where ψ runs over the irreducible $\mathbb{R}\overline{G}$-characters. So

$$2\epsilon^G_{G,i}(\Lambda) = \sum_{\psi \in \mathrm{Irr}(\mathbb{R}\overline{G})} \psi(i) \sum_{\rho \in A_\psi} \mathrm{par}(\ell(\rho)) \ .$$

Meanwhile, we have a direct sum of $\mathbb{R}\overline{G}$-modules $M^{O(G)} = \bigoplus_\psi M_\psi$, where M_ψ is the sum of the $\mathbb{R}\overline{G}$-modules with character ψ. We claim that A_ψ is a triangulation of $S(M_\psi)$. To demonstrate the claim, we shall make use of the admissibility of K as a CG-complex. We have $M_\psi = M^{G_\psi}$, where G_ψ be the index 2 subgroup of CG such that if $\psi(i) = 1$ then $i \in G_\psi \not\ni -i$, otherwise $i \notin G_\psi \ni -i$. But A_ψ is precisely the set of simplexes in K that are fixed by G_ψ. By the admissibility of K as a CG-complex, A_ψ is a triangulation of $S(M^{G_\psi})$. The claim is established. Therefore

$$\sum_{\rho \in A_\psi} \mathrm{par}(\ell(\rho)) = \chi(S(M_\psi)) = 1 - \mathrm{par}(\dim_{\mathbb{R}}(M_\psi)) \ .$$

We have shown that $\epsilon^{G,i}_G(\Lambda) = \sum_{\psi \in \mathrm{Irr}_M(\mathbb{R}G)} \psi(i)$, as required. □

We need to introduce a suitable ghost ring. As a subring of $\mathbb{Q}B_{\mathbb{R}}(G)$, we define

$$\beta_{\mathbb{R}}(G) = \bigoplus_{(I,i)} \mathbb{Z}\, e^G_{I,i}$$

where, as usual, (I, i) runs over representatives of the G-conjugacy classes of C-subelements of G. To distinguish $\beta_{\mathbb{R}}(G)$ from other ghost rings that are sometimes considered in other contexts, let us call $\beta_{\mathbb{R}}(G)$ the **full ghost ring** associated with $B_{\mathbb{R}}(G)$. We have $B_{\mathbb{R}}(G) \leq \beta_{\mathbb{R}}(G) < \mathbb{Q}B_{\mathbb{R}}(G)$, and an element $x \in \mathbb{Q}B_{\mathbb{R}}(G)$ belongs to $\beta_{\mathbb{R}}(G)$ if and only if each $\epsilon^G_{I,i}(x) \in \mathbb{Z}$. Let us mention that $\beta_{\mathbb{R}}(G)$ can be characterized in various other ways: as the $\mathbb{Z}$-span of the primitive idempotents of $\mathbb{Q}B_{\mathbb{R}}(G)$; as the integral closure of $B_{\mathbb{R}}(G)$ in $\mathbb{Q}B_{\mathbb{R}}(G)$; as the unique maximal subring of $\mathbb{Q}B_{\mathbb{R}}(G)$ that is finitely generated as a $\mathbb{Z}$-module.

Since $\epsilon^H_{I,i}(\mathrm{res}_{H,G}(x)) = \epsilon^G_{I,i}(x)$ for all $I \leq H \leq G$, the rings $\beta_{\mathbb{R}}(G)$ combine to form a restriction functor $\beta_{\mathbb{R}}$. Let us mention that, by [Bar04, 5.4, 5.5], $\beta_{\mathbb{R}}$ commutes with induction as well as restriction and isogation, so we can regard $\beta_{\mathbb{R}}$ as a Mackey functor defined on the class of all finite groups. In fact, some unpublished results of Boltje and Olcay Coşkun imply that $\beta_{\mathbb{R}}$ is a biset functor. Let $\beta^{\times}_{\mathbb{R}}(G)$ denote the unit group of $\beta_{\mathbb{R}}(G)$. We have $B^{\times}_{\mathbb{R}}(G) \leq \beta^{\times}_{\mathbb{R}}(G)$, and $x \in \beta^{\times}_{\mathbb{R}}(G)$ if and only if each $\epsilon^G_{I,i}(x) \in C$. For the same reason as before, $\beta^{\times}_{\mathbb{R}}$ is a restriction functor. Actually, part of [Bar04, 9.6] says that $\beta^{\times}_{\mathbb{R}}$ is a Mackey functor.

Lemma 3.3. *Let x be an element of $\mathbb{Z}_{(2)}B_{\mathbb{R}}(G)$ such that $\epsilon^G_{I,i}(x) \equiv_2 \epsilon^G_{I,j}(x)$ for all $I \leq G$ and $i, j \in I$. Write $\lim(x)$ to denote the idempotent of $\beta(G)$ such that $\epsilon^G_I(\lim(x)) \equiv_2 \epsilon^G_{I,i}(x)$. Then $\lim(x) \in \mathbb{Z}_{(2)}B(G)$.*

Proof. For any sufficiently large positive integer m, we have $2^m\mathbb{Z}_{(2)}\beta_{\mathbb{R}}(G) \subseteq \mathbb{Z}_{(2)}B_{\mathbb{R}}(G)$. Choose and fix such m. Let z be the element of $\mathbb{Z}_{(2)}\beta_{\mathbb{R}}(G)$ such that $\lim(x) = x + 2z$. Then

$$\lim(x) = \lim(x)^{2^n} = x^{2^n} + \sum_{j=1}^{2^n} \binom{2^n}{j} 2^j z^j x^{2^n-j}$$

for all positive integers n. When n is sufficiently large, 2^m divides all the binomial coefficients indexed by integers j in the range $1 \leq j \leq m-1$. Choose and fix such n. Then $\lim(x) - x^{2^n}$ belongs to the subset $2^m\mathbb{Z}_{(2)}\beta_{\mathbb{R}}(G)$ of $\mathbb{Z}_{(2)}B_{\mathbb{R}}(G)$. Therefore $\lim(x) \in \mathbb{Z}_{(2)}B_{\mathbb{R}}(G)$. But $\lim(x)$ also belongs to $\mathbb{R}B(G)$, and the required conclusion now follows from Remark 2.1. □

The rationale for the notation $\lim(x)$ is that, under the 2-adic topology, $\lim(x) = \lim_n x^{2^n}$.

We now turn to the reduced Boltje morphism $\overline{\mathrm{bol}}$, which we defined in Section 1. Note that $\overline{\mathrm{bol}}$ can be regarded as the modulo 2 reduction of $\mathrm{bol}^{\{\pm1\},\mathbb{R}}$ because

$$\epsilon^G_{I,i}(\mathrm{bol}^{\{\pm1\},\mathbb{R}}_G[M]) = \chi_I(i) \equiv_2 \dim(M^{O(I)})$$

where χ_I is the $\mathbb{R}I$-character of $M^{O(I)}$.

Theorem 3.4. *Still putting $C = \{\pm1\}$ and letting M be an $\mathbb{R}G$-module, then*

$$\overline{\mathrm{bol}}_G[M] = 1 - 2\lim(\Lambda_{CG}(M))\ .$$

Furthermore, $\lim(\Lambda_{CG}(M)) \in \mathbb{Z}_{(2)}B(G)$ and $\overline{\mathrm{bol}}_G[M] \in \beta^{\times}_{(2)}(G)$.

Proof. By Theorem 3.2, $\epsilon^G_{I,i}(\Lambda_{CG}(M)) \equiv_2 \dim_{\mathbb{R}}(M^{O(I)})$ for any C-subelement (I, i). So the expression $\lim(\Lambda_{CG}(M))$ makes sense and the asserted equality holds. The rider follows from Lemma 3.3. $\square$

The proof of Theorem 1.1 is complete. As an aside, it is worth recording the following description of $\overline{\mathrm{die}}_G[M]$ in terms of monomial Lefschetz invariants of M and $M \oplus \mathbb{R}$, where $\mathbb{R}$ denotes the trivial $\mathbb{R}G$-module.

Corollary 3.5. *Still putting $C = \{\pm 1\}$ and letting M be an $\mathbb{R}G$-module, then*

$$\overline{\mathrm{die}}_G[M] = \Lambda_{CG}(M \oplus \mathbb{R}) - \Lambda_{CG}(M) .$$

Proof. Let $\Lambda = \Lambda_{CG}(M)$ and $\Gamma = \Lambda_{CG}(M \oplus \mathbb{R})$. In the notation of Theorem 3.2,

$$\epsilon^G_{I,i}(\Gamma - \Lambda) = \begin{cases} 1 & \text{if the trivial } \mathbb{R}I\text{-module has odd multiplicity in } (M \oplus \mathbb{R})^{O(I)}, \\ -1 & \text{if the trivial } \mathbb{R}I\text{-module has odd multiplicity in } M^{O(I)}, \end{cases}$$

$$= \begin{cases} 1 & \text{if the trivial } \mathbb{R}I\text{-module has odd multiplicity in } M \oplus \mathbb{R}, \\ -1 & \text{if the trivial } \mathbb{R}I\text{-module has odd multiplicity in } M, \end{cases}$$

$$= \mathrm{par}(\dim_{\mathbb{R}}(M^I)) = \epsilon^G_I(\overline{\mathrm{die}}[M]) .$$

Since this is independent of i, we have $\Gamma - \Lambda \in B(G)$ and $\epsilon^G_I(\Gamma - \Lambda) = \epsilon^G_I(\overline{\mathrm{die}}[M])$. $\square$

4 Dimensions of subspaces fixed by subgroups

We shall prove Theorem 1.2, we shall show that Theorem 1.2 implies Theorem 1.3 and we shall also give a more direct proof of Theorem 1.3.

Let us begin with a direct proof of a special case of Theorem 1.2.

Theorem 4.1. *If G is a 2-group, then* $\dim(M^{O(I)}) \equiv_2 \dim(M)$ *for any $\mathbb{R}G$-module M and any subgroup $I \leq G$.*

Proof. First assume that G has a cyclic subgroup A such that $|G : A| \leq 2$. Letting $x = \overline{\mathrm{die}}_G[M]$, then $\epsilon^G_I(x) = \mathrm{par}(\dim(M^I))$, and we are to show that $\epsilon^G_{O(I)}(x) = \epsilon^G_1(x)$. Our assumption implies that one of the following holds: G is trivial; $O(I) = A < G$ and G is cyclic; $O(I) < A$. By dealing with each of those three possibilities separately, it is easy to see that $O(I)$ is cyclic with generator t^2 for some $t \in G$. A special case of Theorem 3.1 asserts that the function $G \ni g \mapsto \epsilon^G_{\langle g \rangle}(x)/\epsilon^G_1(x) \in \{\pm 1\}$ is a group homomorphism. Therefore $\epsilon^G_{O(I)}(x)/\epsilon^G_1(x) = (\epsilon^G_{\langle t \rangle}(x)/\epsilon^G_1(x))^2 = 1$. The assertion is now established in the special case of the assumption.

For the general case, we shall argue by induction on $|G|$. We may assume that M is simple. Let us recall some material from [Bar06], restating only those conclusions that we need, and only in the special cases that we need. A finite 2-group is called a **Roquette 2-group** provided every normal abelian subgroup is cyclic. A well-known result of Peter Roquette asserts that those 2-groups are precisely as follows: the cyclic 2-groups, the generalized quaternion 2-groups with order at least 8, the dihedral 2-groups with order at least 16, the semidihedral 2-groups with order at least 16. Part of the Genotype Theorem [Bar06, 1.1] says that the simple $\mathbb{R}G$-module M can be written as an induced module $M = \mathrm{Ind}_{G,H}(S)$, where S is a simple $\mathbb{R}H$-module and $H/\mathrm{Ker}(S)$ is a Roquette 2-group.

If M is not absolutely simple, then the $\mathbb{C}G$-module $\mathbb{C} \otimes_{\mathbb{R}} M$ is the sum of two conjugate simple $\mathbb{C}G$-modules, hence each $\dim(M^{O(I)})$ is even and the required conclusion is trivial. So we may assume that M is absolutely simple. Then S must be absolutely simple too.

Suppose that $H = G$. If M is not faithful, then the required conclusion follows from the inductive hypothesis. If M is faithful, then G is a Roquette 2-group. By Roquette's classification, every Roquette 2-group has a cyclic subgroup with index at most 2, and we have already dealt with that case.

So we may assume that $H < G$. Let J be a maximal subgroup of G containing H and let $T = \mathrm{Ind}_{J,H}(S)$. The $\mathbb{R}J$-module T is absolutely simple because $M = \mathrm{Ind}_{G,J}(T)$. Let $x \in G - J$.

Suppose that $\dim(T) = 1$. Then the kernel $N = \mathrm{Ker}(T)$ has index at most 2 in J, so the kernel $N \cap {}^xN = \mathrm{Ker}(M)$ has index at most 2 in N and at most 8 in G. Moreover, if $\mathrm{Ker}(M) \neq N$ then $G/\mathrm{Ker}(M)$ is non-abelian. Replacing G with $G/\mathrm{Ker}(M)$, we reduce to the case where either $|G| = 2$ or else $|G| = 4$ or else G is non-abelian and $|G| = 8$. Any such G has a cyclic subgroup with index at most 2 and, again, the argument is complete in this case.

So we may assume that $\dim(T) \geq 2$. We shall deduce that $\dim(M^{O(I)})$ is even for all $I \leq G$. Identifying T with the subspace $1 \otimes T$ of M, we have $M = T \oplus xT$ as a direct sum of two simple $\mathbb{R}J$-modules. Noting that $O(I) \leq O(G) \leq J$, we have

$$M^{O(I)} = T^{O(I)} \oplus (xT)^{O(I)}$$

as a direct sum of real vector spaces. We are to show that

$$\dim(T^{O(I)}) \equiv_2 \dim((xT)^{O(I)}) \, .$$

If $I \leq J$, then $\dim(T^{O(I)}) \equiv_2 \dim(T) = \dim(xT) \equiv_2 \dim((xT)^{O(I)})$ because, by the inductive hypothesis, the assertion holds for J. Finally, suppose that $I \not\leq J$, in other words, $IJ = G$. The conjugation action of x^{-1} on J induces a transport of structure whereby $O(I)$ is sent to $x^{-1}O(I)x$ and the isomorphism class of xT is sent to the isomorphism class of T. Therefore $\dim((xT)^{O(I)}) = \dim(T^{x^{-1}O(I)x})$. But the element $x \in G - J$ was chosen arbitrarily and, since $IJ = G$, we may insist that $x \in I$, whereupon $x^{-1}O(I)x = O(I)$ and $\dim((xT)^{O(I)}) = \dim(T^{O(I)})$. □

We shall be needing the following result of Tornehave [Tor84]. Another proof of it was given by Yalçın [Yal05, 1.1].

Theorem 4.2. (Tornehave) *Supposing that G is a 2-group, then the reduced tom Dieck map $\overline{\mathrm{die}}_G : A_{\mathbb{R}}(G) \to B^{\times}(G)$ is surjective.*

In view of Theorem 4.2, we see that Theorem 1.3 is equivalent to Theorem 4.1. Our direct proof of Theorem 1.3 is complete.

We mention another way of expressing Theorem 1.3. Let $\overline{\mathrm{sgn}} : B^{\times} \to \beta_{(2)}$ be the unique restriction morphism such that, for any finite group G, the coordinate map $\overline{\mathrm{sgn}}_G$ has image $\overline{\mathrm{sgn}}_G(B^{\times}) = \{\pm 1_{B(G)}\}$. Thus, $\epsilon_I^G(\overline{\mathrm{sgn}}(x)) = \epsilon_1^G(x)$ for all $I \leq G$ and $x \in B^{\times}(G)$. Plainly, Theorem 1.3 can be expressed as follows.

Theorem 4.3. *As restriction functors for the class of finite 2-groups, $\overline{\mathrm{bol}} = \overline{\mathrm{sgn}} \circ \overline{\mathrm{die}}$.*

We now turn towards the task of proving Theorem 1.2. The following theorem of Andreas Dress can be found in, for instance, Benson [Ben91, 5.4.8]. Let p be a prime. We write $\mathbb{Z}_{(p)}$ for the ring of p-local integers. We write $O^p(G)$ for the largest normal subgroup of G such that $G/O^p(G)$ is a p-group. Recall that G is said to be ***p*-perfect** provided $G = O^p(G)$.

Theorem 4.4. (Dress) *Given a prime p and an idempotent $y \in \mathbb{Q}B(G)$, then $y \in \mathbb{Z}_{(p)}B(G)$ if and only if $\epsilon^G_I(y) = \epsilon^G_{O^p(I)}(y)$ for all $I \leq G$. In particular, the condition $\epsilon^G_H(y) = 1$ characterizes a bijective correspondence between the primitive idempotents y of $\mathbb{Z}_{(p)}B(G)$ and the conjugacy classes of p-perfect subgroups H of G.*

The next corollary is worth mentioning, although it yields no constraints on the units of $B(G)$ and it will not be used below.

Corollary 4.5. *Given $x \in \mathbb{Z}_{(2)}B(G)$, then $\epsilon^G_I(x) \equiv_2 \epsilon^G_{O^2(I)}(x)$ for all $I \leq G$.*

Proof. The hypothesis on x implies that $\epsilon^G_{I,i}(x) = \epsilon^G_{I,j}(x)$ for all $I \leq G$ and all $i, j \in I$. By Lemma 3.3 and Theorem 4.3, $\epsilon^G_I(x) \equiv_2 \epsilon^G_I(\lim(x)) = \epsilon^G_{O^2(I)}(\lim(x)) \equiv_2 \epsilon^G_{O^2(I)}(x)$. □

Putting $C = \{\pm 1\}$ and letting M be an $\mathbb{R}G$-module, Theorems 3.4 and 4.3 yield

$$\dim(M^{O(I)}) \equiv_2 \epsilon^G_I(\Lambda_{CG}(M)) = \epsilon^G_{O^2(I)}(\Lambda_{CG}(M)) \equiv_2 \dim(M^{O(O^2(I))}) = \dim(M^{O^2(I)}) \,.$$

The proof of Theorem 1.2 is complete.

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