Quizzes, with solutions

MATH 224, *Linear Algebra 2*, Spring 2024, Laurence Barker

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Quiz 1: Let V be a 2-dimensional vector space over the field $\mathbb{F}_5 = \mathbb{Z}/5$ of order 5. How many subspaces does V have?

Solution: The unique 0-dimensional subspace of V is $\{0\}$. The unique 2-dimensional subspace of V is V. All the other subspaces are 1-dimensional. Each of the $5^2 - 1 = 24$ nonzero vectors in V generates a 1-dimensional subspace. On the other hand, each of the 1-dimensional subspace is generated by any of its $5 - 1 = 4$ elements. So the number of 1-dimensional subspaces is $24/4 = 6$, and the total number of subspaces of V is $1 + 1 + 6 = 8$.

Quiz 2: Consider the coding scheme over $\mathbb{F}_2$ with generating matrix $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) Write out the decoding table with syndromes.
(b) For the received word 110, what is the syndrome and the decoded codeword?

Solution: Part (a). The Hamming matrix is $H = [1\ \ 1\ \ 1]$. The decoding table is as shown.

00	01	10	11	syndrome
000	011	101	110	0
001	010	100	111	1

Part (b). The decoded codeword is 110 and the decoded message word is 11.

Quiz 3: Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

That is to say, find an invertible matrix P and a diagonal matrix D such that $D = PDP^{-1}$.

Solution: The characteristic polynomial of A is

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (3-\lambda)(1-\lambda)$$

which has roots 3 and 1. The eigenvalues 3 and 1 have associated eigenvectors $(1, 1)$ and $(1, -1)$, respectively. So we can put

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} , \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} .$$

Comment 1: As a check, we note that

$$PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = A .$$

Comment 2: We made use of the general fact that, for a diagonalizable $n \times n$ matrix A with a basis of eigenvectors f_1, ..., f_n and corresponding eigenvalues λ_1, ..., λ_n, we have $A = PDP^{-1}$ where the (j, j) entry of D is λ_j and the j-th column of P is the column vector

$$f_j = \begin{bmatrix} p_{1,j} \\ \vdots \\ p_{i,j} \\ \vdots \\ p_{n,j} \end{bmatrix} .$$

Quiz 4: Evaluate A^4 where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution: We have $A^2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, hence $A^4 = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

Quiz 5: Let $\langle - | - \rangle$ be a symmetric bilinear form on a real vector space V auch that $\langle - | - \rangle$ is nonzero in the sense that $\langle x \,|\, y \rangle \neq 0$ for some $x, y \in V$. Show that $\langle x \,|\, x \rangle \neq 0$ for some $x \in V$.

Solution: Let $y, z \in V$ such that $\langle y \,|\, z \rangle \neq 0$. We may assume that $\langle y \,|\, y \rangle = \langle z \,|\, z \rangle = 0$. Putting $x = y + z$, then $\langle x \,|\, x \rangle = 2\langle y \,|\, z \rangle \neq 0$.