# Quizzes, with solutions

MATH 224, *Linear Algebra 2*, Spring 2024, Laurence Barker

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**Quiz 1:** Let $V$ be a 2-dimensional vector space over the field $\mathbb{F}_5 = \mathbb{Z}/5$ of order 5. How many subspaces does $V$ have?

*Solution:* The unique 0-dimensional subspace of $V$ is $\{0\}$. The unique 2-dimensional subspace of $V$ is $V$. All the other subspaces are 1-dimensional. Each of the $5^2 - 1 = 24$ nonzero vectors in $V$ generates a 1-dimensional subspace. On the other hand, each of the 1-dimensional subspace is generated by any of its $5 - 1 = 4$ elements. So the number of 1-dimensional subspaces is $24/4 = 6$, and the total number of subspaces of $V$ is $1 + 1 + 6 = 8$.

**Quiz 2:** Consider the coding scheme over $\mathbb{F}_2$ with generating matrix $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

**(a)** Write out the decoding table with syndromes.
**(b)** For the received word 110, what is the syndrome and the decoded codeword?

*Solution:* Part (a). The Hamming matrix is $H = [1\ \ 1\ \ 1]$. The decoding table is as shown.

| 00 | 01 | 10 | 11 | syndrome |
|---|---|---|---|---|
| 000 | 011 | 101 | 110 | 0 |
| 001 | 010 | 100 | 111 | 1 |

Part (b). The decoded codeword is 110 and the decoded message word is 11.

**Quiz 3:** Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

That is to say, find an invertible matrix $P$ and a diagonal matrix $D$ such that $D = PDP^{-1}$.

*Solution:* The characteristic polynomial of $A$ is

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (3-\lambda)(1-\lambda)$$

which has roots 3 and 1. The eigenvalues 3 and 1 have associated eigenvectors $(1, 1)$ and $(1, -1)$, respectively. So we can put

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} , \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} .$$

*Comment 1:* As a check, we note that

$$PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = A .$$

*Comment 2:* We made use of the general fact that, for a diagonalizable $n \times n$ matrix $A$ with a basis of eigenvectors $f_1$, ..., $f_n$ and corresponding eigenvalues $\lambda_1$, ..., $\lambda_n$, we have $A = PDP^{-1}$ where the $(j, j)$ entry of $D$ is $\lambda_j$ and the $j$-th column of $P$ is the column vector

$$f_j = \begin{bmatrix} p_{1,j} \\ \vdots \\ p_{i,j} \\ \vdots \\ p_{n,j} \end{bmatrix} .$$

**Quiz 4:** Evaluate $A^4$ where $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

*Solution:* We have $A^2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, hence $A^4 = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

**Quiz 5:** Let $\langle - | - \rangle$ be a symmetric bilinear form on a real vector space $V$ auch that $\langle - | - \rangle$ is nonzero in the sense that $\langle x \,|\, y \rangle \neq 0$ for some $x, y \in V$. Show that $\langle x \,|\, x \rangle \neq 0$ for some $x \in V$.

*Solution:* Let $y, z \in V$ such that $\langle y \,|\, z \rangle \neq 0$. We may assume that $\langle y \,|\, y \rangle = \langle z \,|\, z \rangle = 0$. Putting $x = y + z$, then $\langle x \,|\, x \rangle = 2\langle y \,|\, z \rangle \neq 0$.

**Quiz 6:** Let $H_n$ denote the set of Hermitian $n{\times}n$ matrices.

**(a)** Show that, given $A, B \in H_n$, then $AB \in H_n$.

**(b)** Is $H_n$ a subspace of the complex vector space $\mathrm{Mat}_n(\mathbb{C})$?

*Solution:* Part (a). We have $\langle ABx \,|\, y \rangle = \langle Bx \,|\, Ay \rangle = \langle x \,|\, BAy \rangle = \langle x \,|\, ABy \rangle$ for all $x, y \in \mathbb{C}^n$.

Part (b). No. Indeed, the identity $n{\times}n$ matrix $I_n$ is Hermitian but $iI$ is not Hermitian. Thus, $H_n$ is not closed under scalar multiplication.

**Quiz 7:** A real $n{\times}n$ matrix $A$ is said to be:

- **symmetric** provided $A$ is Hermitian, in other words, $A^T = A$,
- **orthogonal** provided $A$ is unitary, in other words, $A^T = A^{-1}$.

**(a)** Do the symmetric matrices in $\mathrm{Mat}_n(\mathbb{R})$ comprise a subspace of $\mathrm{Mat}_n(\mathbb{R})$?

**(b)** Do the orthogonal matrices in $\mathrm{Mat}_n(\mathbb{R})$ comprise a subspace of $\mathrm{Mat}_n(\mathbb{R})$?

*Solution:* Part (a). Yes, since the symmetric matrices are closed under addition and scalar multiplication.

Part (b). No, because the zero matrix is not an orthogonal matrix.

*Comment:* Examiners tend to like scripts that answer any yes/no question with a clear "yes" or a clear "no".

**Quiz 8:** Recall that $\cosh(t) = (e^t + e^{-t})/2$ and $\sinh(t) = (e^t - e^{-t})/2$ for $t \in \mathbb{R}$. Diagonalize the real symmetric matrix

$$H_t = \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix} .$$

*Solution:* Write $c = \cosh(t)$ and $s = \sinh(t)$. Since $\mathrm{tr}(H_t) = 2c$ and $\det(H_t) = c^2 - s^2 = 1$, the characteristic polynomial of $H_t$ is $\det(H_t - \lambda I) = \lambda^2 - 2c\lambda + 1$. The eigenvalues of $H_t$ are the roots $e^t$ and $e^{-t}$ of the polynomial, and the associated eigenvectors are $(1,1)$ and $(-1,1)$, respectively. So $H_t = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} , \qquad D = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} .$$

*Comment:* A standard theorem tells us that any real symmetric matrix can be expressed in the form $PDP^{-1}$ where $P$ is an orthogonal matrix and $D$ is a real diagonal matrix. Since the eigenvalues $e^t$ and $e^{-t}$ have associated eigenvectors $(1,1)/\sqrt{2}$ and $(-1,1)/\sqrt{2}$, respectively, we could have put

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

which is an orthogonal matrix.

**Quiz 9:** Let $A$ be an orthogonal operator on $\mathbb{R}^n$. Let $x$ be a real eigenvector of $A$. Let $\lambda$ be the eigenvalue of $A$ associated with $x$. What are the possible values of $\lambda$?

*Solution:* Since the $\mathbb{C}$-linear extension of $A$ is a unitary operator on $\mathbb{C}^n$, we have $|\lambda| = 1$. But $\lambda \in \mathbb{R}$. Therefore $\lambda = \pm 1$. Plainly, both 1 and $-1$ can arise as eigenvalues of othogonal operators on $\mathbb{R}^n$. In conclusion, the possible values of $\lambda$ are 1 and $-1$.