Quizzes, with solutions

MATH 220, *Linear Algebra*, Spring 2024, Laurence Barker

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Quiz 1: By Gaussian elimination, solve $x + y + z = 6$, $x + 2y + 4z = 11$, $x + 3y + 9z = 18$.

Solution: The augmented matrix for the system of linear equations is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 11 \\ 1 & 3 & 9 & 18 \end{array}\right]$.

Subtracting row 1 from the other two rows yields $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 8 & 12 \end{array}\right]$.

Subtracting 2 times row 2 from row 3 yields $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 2 \end{array}\right]$.

Dividing row 2 by 2 gives $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

The equations expressed by this aumented matrix are $x + y + z = 6$, $y + 3z = 5$, $z = 1$. Hence, $y = 5 - 3z = 5 - 3 = 2$ and $x = 6 - y - z = 6 - 2 - 1 = 3$. In conclusion, the solution is $(x, y, z) = (3, 2, 1)$.

Quiz 2: Let $A = (a_{i,j})$ be the 99×99 matrix such that

$$a_{i,j} = \begin{cases} 0 & \text{if } i + j \leq 99, \\ 1 & \text{if } i + j \geq 100. \end{cases}$$

Thus, the entries of A are 0 above the diagonal from bottom left to top right. The entries are 1 on and below that diagonal. Evaluate $\det(A)$.

Solution: Applying 49 transpositions of rows, we obtain a matrix in row echelon form whose determinant is 1. Each of those row operations multiplies the determinant by -1, so $\det(A) = (-1)^{49} = -1$.

Quiz 3: Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

(a) Find a basis for $\ker(A)$.

(b) Evaluate (A) and $\mathrm{rank}(A)$.

Solution: Since $(1, 1, 1) \in \ker(A)$, so $(A) \geq 1$. Since $\mathrm{rank}(A) + (A) = 3$, we have $\mathrm{rank}(A) \leq 2$. The first 2 columns of A are linearly independent, so $\mathrm{rank}(A) \geq 2$. Therefore $\mathrm{rank}(A) = 2$ and $(A) = 1$. It now follows that $\{(1, 1, 1)\}$ is a basis for $\ker(A)$.

Comment 1: A more routine solution would be first to show, by Gaussian elimination, that the solutions to $Av = 0$ are precisely those $v = (x, y, z)$ such that $x = y = z$. Hence, $\{(1, 1, 1)\}$ is a basis for $\ker(A)$. It follows that $(A) = 1$ and, by the rank-nullity formula, $\mathrm{rank}(A) = 2$.

Comment 2: This is just a style comment, not very important. Some slightly imperfect answers to part (a):

• "So $(1, 1, 1)$ is a basis for $\ker(A)$". How can a vector be a basis? The intention must have been "So $\{(1, 1, 1)\}$ is a basis for $\ker(A)$". Minus one mark.

• "So a basis for $\ker(A) = (1, 1, 1)$". How can $\ker(A)$, which is a subspace, be identical to $(1, 1, 1)$, which is a vector? The intention must have been "So one basis for $\ker(A)$ is $\{(1, 1, 1)\}$." Minus one mark.

At this level of mathematics, such mistakes do not matter very much, because the intention can be easily gleaned. In more advanced work, where more exotic ideas have to be communicated, such mistakes would tend to render the text incomprehensible.

Quiz 4: In $\mathbb{R}^2$, let $u_1 = (1, 2)$ and $u_2 = (2, 3)$. Apply the Gram–Schmidt process to produce an orthonormal basis $\{w_1, w_2\}$.

Solution: We have an orthogonal basis $\{v_1, v_2\}$ given by $v_1 = u_1$ and

$$v_2 = u_2 - \frac{v_1.u_2}{\|v_1\|^2} v_1 \ .$$

Thus, $v_1 = (1, 2)$ and

$$v_2 = (2, 3) - \frac{1.2 + 2.3}{1^2 + 2^2}(1, 2) = (2/5, -1/5) \ .$$

We put $w_1 = v_1/\|v_1\|$ and $w_2 = v_2/\|v_1\|$. The constructed orthonormal basis is

$$\{w_1, w_2\} = \{(1, 2)/\sqrt{5}, (2, -1)/\sqrt{5}\} \ .$$