Quizzes, with solutions

MATH 220, *Linear Algebra*, Spring 2024, Laurence Barker

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Versions of this file, updated as the course progresses, can be found on my homepage.

Quiz 1: By Gaussian elimination, solve $x + y + z = 6$, $x + 2y + 4z = 11$, $x + 3y + 9z = 18$.

Solution: The augmented matrix for the system of linear equations is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 11 \\ 1 & 3 & 9 & 18 \end{array}\right]$.

Subtracting row 1 from the other two rows yields $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 8 & 12 \end{array}\right]$.

Subtracting 2 times row 2 from row 3 yields $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 2 \end{array}\right]$.

Dividing row 2 by 2 gives $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

The equations expressed by this aumented matrix are $x + y + z = 6$, $y + 3z = 5$, $z = 1$. Hence, $y = 5 - 3z = 5 - 3 = 2$ and $x = 6 - y - z = 6 - 2 - 1 = 3$. In conclusion, the solution is $(x, y, z) = (3, 2, 1)$.

Quiz 2: Let $A = (a_{i,j})$ be the 99×99 matrix such that

$$a_{i,j} = \begin{cases} 0 & \text{if } i + j \leq 99, \\ 1 & \text{if } i + j \geq 100. \end{cases}$$

Thus, the entries of A are 0 above the diagonal from bottom left to top right. The entries are 1 on and below that diagonal. Evaluate $\det(A)$.

Solution: Applying 49 transpositions of rows, we obtain a matrix in row echelon form whose determinant is 1. Each of those row operations multiplies the determinant by -1, so $\det(A) = (-1)^{49} = -1$.

Quiz 3: Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

(a) Find a basis for $\ker(A)$.

(b) Evaluate (A) and $\mathrm{rank}(A)$.

Solution: Since $(1, 1, 1) \in \ker(A)$, so $(A) \geq 1$. Since $\mathrm{rank}(A) + (A) = 3$, we have $\mathrm{rank}(A) \leq 2$. The first 2 columns of A are linearly independent, so $\mathrm{rank}(A) \geq 2$. Therefore $\mathrm{rank}(A) = 2$ and $(A) = 1$. It now follows that $\{(1, 1, 1)\}$ is a basis for $\ker(A)$.

Comment 1: A more routine solution would be first to show, by Gaussian elimination, that the solutions to $Av = 0$ are precisely those $v = (x, y, z)$ such that $x = y = z$. Hence, $\{(1, 1, 1)\}$ is a basis for $\ker(A)$. It follows that $(A) = 1$ and, by the rank-nullity formula, $\mathrm{rank}(A) = 2$.

Comment 2: This is just a style comment, not very important. Some slightly imperfect answers to part (a):

• "So $(1, 1, 1)$ is a basis for $\ker(A)$". How can a vector be a basis? The intention must have been "So $\{(1, 1, 1)\}$ is a basis for $\ker(A)$". Minus one mark.

• "So a basis for $\ker(A) = (1, 1, 1)$". How can $\ker(A)$, which is a subspace, be identical to $(1, 1, 1)$, which is a vector? The intention must have been "So one basis for $\ker(A)$ is $\{(1, 1, 1)\}$." Minus one mark.

At this level of mathematics, such mistakes do not matter very much, because the intention can be easily gleaned. In more advanced work, where more exotic ideas have to be communicated, such mistakes would tend to render the text incomprehensible.

Quiz 4: In $\mathbb{R}^2$, let $u_1 = (1, 2)$ and $u_2 = (2, 3)$. Apply the Gram–Schmidt process to produce an orthonormal basis $\{w_1, w_2\}$.

Solution: We have an orthogonal basis $\{v_1, v_2\}$ given by $v_1 = u_1$ and

$$v_2 = u_2 - \frac{v_1.u_2}{\|v_1\|^2} v_1 \ .$$

Thus, $v_1 = (1, 2)$ and

$$v_2 = (2, 3) - \frac{1.2 + 2.3}{1^2 + 2^2}(1, 2) = (2/5, -1/5) \ .$$

We put $w_1 = v_1/\|v_1\|$ and $w_2 = v_2/\|v_1\|$. The constructed orthonormal basis is

$$\{w_1, w_2\} = \{(1, 2)/\sqrt{5}, (2, -1)/\sqrt{5}\} \ .$$

Quiz 5: Find the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution: The characteristic equation is

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 1 = \lambda_2 - 2\lambda \ .$$

The eigenvalues of A are the solutions to the equation, namely 0 and 2.

Comment: Alternatively, by inspection, $(1, -1)$ and $(1, 1)$ are eigenvectors with associated eigenvalues 0 and 2, respectively. Since A is a 2×2 matrix, there cannot be any other eigenvalues. Therefore, the eigenvalues of A are 0 and 2.

Quiz 6: Diagonalize the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, in other words, find invertible P and diagonal D such that $A = PDP^{-1}$.

Solution: Plainly, the eigenvalues of A are 2 and 4, furthermore, $(1,0)$ is a 2-eigenvector. Supposing (x,y) is a 4-eigenvector, then

$$\begin{bmatrix} 2-4 & 3 \\ 0 & 4-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Hence, $2x = 3y$. Putting $y = 1$, we have $x = 3/2$. So $(3/2, 1)$ is a 4-eigenvector. We have shown that we can put

$$P = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix} , \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} .$$

Comment: The solution does not require a check, but let us check it anyway. The above values of P and D yield

$$PDP^{-1} = \begin{bmatrix} 2 & 6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix} = A .$$

Quiz 7: Let A be a real symmetric $n \times n$ matrix with an eigenvector x. Let $y \in \mathbb{R}^n$ such that $\langle x \,|\, y \rangle = 0$. Show that $\langle x \,|\, Ay \rangle = 0$.

Solution: We have $\langle x \,|\, Ay \rangle = \langle Ax \,|\, y \rangle = \langle \lambda Ax \,|\, y \rangle = \lambda \langle x \,|\, y \rangle = 0$ where λ is the eigenvalue associated with x.

Quiz 8: **(a)** Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} .$$

By direct calculation, show that any 2-eigtenvector of A and any 0-eigenvector of A are orthogonal to each other.

(b) Now let A be any real symmetric matrix and let λ_1 and λ_2 be distinct eigenvalues of A. Show that any λ_1-eigenvector of A is orthogonal to any λ_2-eigenvector of A.

Solution: Part (a). This is clear because the 2-eigenvectors have the form (x,x) and the 0-eigenvectors have the form $(x,-x)$ for nonzero $x \in \mathbb{R}$.

Part (b). Let f_1 and f_2 be a λ_1-eigenvector and a λ_2-eigenvector of A, respectivelty. Then

$$\lambda_1 \langle f_1 \,|\, f_2 \rangle = \langle \lambda_1 f_1 \,|\, f_2 \rangle = \langle A f_1 \,|\, f_2 \rangle = \langle f_1 \,|\, A f_2 \rangle = \langle f_1 \,|\, \lambda_2 f_2 \rangle = \lambda_2 \langle f_1 \,|\, f_2 \rangle .$$

So $\langle f_1 \,|\, f_2 \rangle = 0$.

Comment: In the set version of the quiz, I write down the incorrect eigenvalues of the given matrix, and this caused confusion. I take the view that, the intention must have been clear to those who understood the material. So there are no marks in (a) for those who failed to make the necessary correction. (Mathematical text often does have slips, and part of mathematical competence is in making appropriate corrections to the texts that one reads.)