Syllabus for the Bilkent University Mathematics PhD Programme Qualifying Exam

Latest revision: 10 April 2022.

Written Exam: The duration of the exam is three hours. There are four subject areas: Algebra, Analysis, Applied Mathematics, Geometry/Topology. In each subject area, there are four questions. The candidate is to attempt two questions from each subject area. As guides for preparation, the main syllabi topics for the four subject areas can be found on the following pages.

Oral Exam: The candidate is to make a fifteen-minute presentation directed as if towards a general audience. This is to be followed by about fifteen minutes of questions from the Jury. The questions need not be confined to the topic of the presentation. Some of the questions may be requests for pedagogical clarification of basic concepts. The assessment will be for teaching and communication. No credit will be awarded for specialist erudition.

ALGEBRA

Group Theory: Structure Theorem for Fintely Generated Abelian Groups, permutation sets, Sylow's Theorem, Jordan–Hölder Theorem. Familiarity with groups of small order.

Galois Theory: Polynomial rings over fields. Eisenstein's Criterion, Artin's Primitive Element Theorem, Fundamental Theorem of Galois Theory, Unsolvability of the Quintic. Calculation of Galois groups for specific polynomials.

Ring Theory: Applications of Zorn's Lemma. Semisimple, Artinian and Noetherian rings and modules. Jacobson radical. Artin–Wedderburn Structure Theorem.

General bibliography:

- M. Artin, "Algebra", (Prentise–Hall, New Jersey, 1991).
- I. M. Isaacs, "Algebra: a graduate course", (Brookes/Cole, California, 1994).
- B. L. van def Waerden, "A History of Algebra", (Springer, Berlin, 1985).

Specific bibliography:

- E. Artin "Galois Theory", revised by A. N. Milgram (Dover, New York, 1998).
- M. Aschbacher, "Finite group theory", 2nd ed. (C.U.P. 2000).
- T. Y. Lam, "A First Course in Non-Commutative Rings", (Springer, Berlin, 1991).
- W. R. Scott, "Group Theory", (Dover, New York, 1987).

ANALYSIS

Question 1: Measure theory

1. Algebra of sets. Borel sets and Baire functions.
2. Measures and outer measures. Measurable and nonmeasurable sets.
3. Measurable functions. Lusin's and Egorov's theorems. Modes of convergence.
4. Integration. Limit theorems for Lebesgue integral.
5. Differentiation of Lebesgue integral. Functions of bounded variation. Absolutely continuous functions.
6. Spaces of integrable functions.
7. Product measures. Signed measures. The Radon-Nikodym theorem. Hahn, Jordan and Lebesgue decompositions.

Books:

W. Rudin, *Real and Complex Analysis*, McGraw-Hill Int.Editions, 1966
D.L. Cohn, *Measure Theory*, Birkhauser 2013.

Question 2: Functional analysis

1. Metric and topological spaces. Completeness. Compactness. Compactness in concrete function spaces. Continuous mappings on compact metric spaces.
2. Hilbert spaces. Orthogonality. General Fourier series.
3. Banach spaces. Linear functionals and linear operators. Dual spaces. The Hahn-Banach theorem.
4. Category. Baire theorem. The Banach-Steinhaus theorem. The open mapping theorem.
5. Weak Topologies.

Books:

W. Rudin, *Real and Complex Analysis*, McGraw-Hill Int.Editions, 1966.
W. Rudin, *Functional Analysis*, McGraw-Hill Int.Editions, 1973.
J.B. Conway, *A Course in Functional Analysis*, Springer 2003.
E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley, 1989.

Questions 3, 4: COMPLEX ANALYSIS

A. Holomorphic Functions: Complex derivatives, Cauchy-Riemann equations, Cauchy theorem and integral formula, Morera theorem, power series representation, sequences of holomorphic functions, uniqueness theorem, open mapping theorem, maximum modulus principle, winding numbers, Cauchy theorem for multiply connected regions, Cauchy-type integrals.

B. Singularities: Isolated singularities, Laurent series, residue theorem, argument principle, Rouché theorem, residues at infinity, evaluation of integrals and sums.

C. Conformal Mapping: Preservation of angles, Schwarz-Pick lemma, mapping by Möbius transformations, normal families, Riemann mapping theorem, simply connected regions.

D. Harmonic Functions: Maximum principle, mean value property, Poisson integral, Harnack theorem.

Content

The exams contents are given in 5 **equivalent** ways for each of the 5 references below. To repeat, it is enough to study from only **1** book, say [**U**].

[**C**] Chapters I, III, IV, V, sections VI.1–VI.2, VII.2, VII.4, VIII.2, X.1–X.2.

[**G**] Chapters I, Ii, sections III.1–III.5, chapter IV, sections V.1–V.7, VI.1–VI.4, chapter VII, sections VIII.1–VIII.4, VIII.6–VIII.8, IX.1–IX.2, chapter X, XI.1–XI.2, XI.5–XI.6.

[**GK**] Chapters 1, 2, 3, 4, 5, 6, sections 7.1–7.6, 11.1–11.4

[**R**] Chapter 10, sections 11.1–11.3, 12.1–12.2, 13.4, 14.1–14.4. (This book does not have section numbers, but 12.1, for example, refers to the first unnumbered section in chapter 12 which is titled Introduction, etc.)

[**U**] Chapters 0, 1, 2, 3, 4, 5, 7, 8, sections 9.3–9.5, 10.0–10.4, 10.7, chapter 11.

References

[**C**] J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer, 1978.
[**G**] T. W. Gamelin, *Complex Analysis*, Springer, 2001.
[**GK**] R. E. Greene & S. G. Krantz, *Function Theory of One Complex Variable*, 3rd ed., AMS, 2006.
[**R**] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, 1987.
[**U**] D. C. Ullrich, *Complex Made Simple*, AMS, 2008.

APPLIED MATHEMATICS

Questions 1, 2: MATH 543 Methods of Applied Mathematics I

1. Function spaces, completeness, square integrable functions. Orthogonal sets of vectors and the Bessel inequality. Basis and Parseval's relation. Weierstrauss' theorem. Classification of orthogonal polynomials, classical orthogonal polynomials. Trigonometric series, generalized functions

2. Second order differential equations. Fundamental solutions. method of variation of constants. Generalized Green's identity: Adjoint operators and adjoint boundery conditions. The method of Green's function. The Sturm–Liouville problem, classification of singular points. The Frobenius method: The series solutions of linear DEs. Fuchsian differential equations, the hypergeometric function. Solutions of DEs by integral representations. Integral representations of hypergeometric functions.

3. Regular perturbations, Poincare–Lindstedt method. Singular purturbations, boundery layer problems. WKB approxinmation, asymptotic expansion of integrals.

4. Calculus of variations, necessary condition, Euler–Lagrange equations, Lagrange functions depending on higher derivatives, null Lagrange functions, lagrange function of given DE. Lagrange function with several dependent variables. Iso-perimetric problems.
Books: Dennery and Krzywicki for 1 and 2. Logan for 3. Logan and Hildebrand for 4.
For more details see the course web site
http:// www.fen.bilkent.edu.tr/gurses/applied1.html.
For Qualifying Exam questions, see those *assigned exercises* at this web site that are related to the above subjects.

Questions 3, 4: MATH 544 Methods of Applied Mathematics II

1. Partial differential equation models (G–L, Copson, Logan). Classification of 2nd order partial differential equations (G–L, Copson) The Cauchy–Kowalewsky theorem (Copson). Linear, quasi-linear, half-linear equations (Copson). Initial and BV problems for the wave equation (G–L). Some existence and uniquenessn theorems (G–L). First order hyperbolic systems. The Riemann Method (G–L, Copson). Exact solutions, uniqueness, maximum-minimum theorems of hyperbolic type of (the wave) equation, of paranolic type of (the heat) equation, of elliptic type of (the Laplace) equation (G–L, D–K).

2. Integral equations and Green's functions (G–L, Hildebrand). Sturm–Liouville problems, Neumann series, Hilbert–Schmidt theory of singular integral equations.

3. Stability and bifurcation (Logan, Boyce–DiPrima Ch. 9). One-dimensional problems, two-dimensional problems. Hydrodynamics stability.

DK = Dennery and Krzywicki, G–L = Roland B. Guenter and J. W. Lee.
For more details see the web site
http:// www.fen.bilkent.edu.tr/gurses/applied2.html.
For Qualifying Exam questions, see those *assigned exercises* at this web site that are related to the above subjects.

GEOMETRY AND TOPOLOGY

Questions 1, 2: Algebraic Geometry

Affine varieties, projective varieties, morphisms, rational maps, blow-ups, resolution of simplest singularities, nonsingular varieties, nonsingular curves, intersections in projective space. [Hartshorne, chapter I]

The Riemann-Hurwitz formula, The Riemann-Roch theorem for compact complex Riemann surfaces and its application to low genus curves, Abel's theorem and its applications. [Griffiths, chapters III, IV, V]

Relevant courses: MATH 430, 431, 505, 591, 633, 634.

References:
Hartshorne, Algebraic Geometry, Springer-Verlag, 1977.
Griffiths, Introduction to algebraic curves, AMS, 1989.

Question 3: General Topology

1. Topological spaces and continuous functions: Topological spaces, basis and subbasis, subspace topology, continuous functions, product topology, metric topology, quotient topology, topological groups.

2. Compactness: Compact spaces, compact sets in $\mathbb{R}^n$, Heine–Borel Theorem, Paracompact spaces, Tychonoff Theorem (finite product version), limit-point compactness, sequential compactness, compactness in metric spaces, local compactness and one-point compactification.

3. Connectedness: Connected spaces, path-connected spaces, components, local connectedness, local path-connectedness.

4. Separation and Countability Properties: T_0, Hausdorff, regular, normal spaces, partitions of unity, countability properties; Lindelöf, separable, countably compact spaces

References:
Munkres J., *Topology, a First Course*, (sections: 2.1-2.10, 3.1-3.8, 4.1- 4.3)
Bredon G.E., *Geometry and Topology*, Chapter 1.
Willard S., *General Topology*, (sections: 2, 3, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17, 18, 19. In section 19 only one-point compactification is included.)

Question 4: Algebraic Topology:

Fundamental group, Van Kampen's Theorem, Covering spaces. Homotopy groups. Singular homology: Homotopy invariance, homology long exact sequence, Mayer- Vietoris sequence, excision. Cellular homology. Homology with coefficients. Simplicial homology and the equivalence of simplicial and singular homology. Axioms of homology. Homology and fundamental group. Simplicial approximation. Cohomology of spaces, Universal Coefficient Theorem, Cup product, Künneth formula. Topological Manifolds.

Main Reference: A. Hatcher, *Algebraic Topology* (2000).

Other References: J. R. Munkres, *A First Course in Topology* (Chapter 8).
J. R. Munkres, *Elements of Algebraic Topology.*

G. Bredon, *Geometry and Topology.*
J. J. Rotman, *An Introduction to Algebraic Topology.*
E. Spanier, *Algebraic Topology.*
M. Greenberg and J. Harper. *Algebraic Topology.*
W. S. Massey, *A Basic Course in Algebraic Topology.*