BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

27 September 2022

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3.

Examiner 2: Commutative Algebra, question A4.

Examiner 3: Real Analysis, questions B1, B2.

Examiner 4: Complex Analysis, questions B3, B4.

Examiner 5: Methods of Applied Mathematics, questions C1, C2, C3, C4.

Examiner 6: Geometry and Topology, questions D1, D2.

Examiner 7: Geometry and Topology, questions D3, D4.

Time allowed: three hours.

Section A: Algebra

A1: Classify, up to isomorphism, the groups G such that the automorphism group $\mathrm{Aut}(G)$ is trivial. (Warning: few marks will be awarded for dealing only with the finite case.)

A2: Let E be the splitting field for $X^3 - X^2 - 3$ over $\mathbb{Q}$.

(a) Find the Galois group of the field extension $E/\mathbb{Q}$.

(b) How many fields L are there such that $\mathbb{Q} \leq L \leq E$?

A3: Show that every automorphism of a field F extends to an automorphism of the algebraic closure of F.

A4: Let K be a field. Let $n \geq 2$ and let A the algebra of upper triangular $n \times n$-matrices over K. Show that, up to isomorphism:

(a) There are only finitely many simple A-modules.

(b) There are infinitely many indecomposable A-modules.

Section B: Analysis

B1: Suppose $1 \leq p < q \leq \infty$.

(a) Show that $f = (f_n)_{n=1}^{\infty} \in l_p$ implies $f \in l_q$ and $||f||_q \leq ||f||_p$.

(b) Show that $f = f(x) \in L_q[0,1]$ implies $f \in L_p[0,1]$ and $||f||_p \leq ||f||_q$.

(c) Present $f \in L_q(\mathbb{R})$ such that $f \notin L_r(\mathbb{R})$ for each r with $1 \leq r \leq \infty, r \neq q$.

B2: Let $X = \{(x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} \frac{|x_n|}{n} < \infty\}, ||x|| = \sup_n \frac{|x_n|}{n^2}$. Show that the norm space $(X, ||\cdot||)$ is not complete and find its completion.

B3: Let $V = \mathbb{C} \backslash [-1,1]$ and $f(z) = z^2 - 1$. Prove that f does not have a holomorphic logarithm on V.

B4: Let f be a conformal (one-to-one, onto, and biholomorphic) map of the open unit disc $\mathbb{D}$ onto a domain G. Let also $c = f(0)$ and d be the distance of c to the boundary ∂G of G. Show that $d \leq |f'(0)|$.

Section C: Methods of Applied Mathematics

C1: Solve the following boundary value problem by the use of Greens function

$$u_{xx} = f(x), \quad x \in (0,1),$$
$$u(0) - \alpha\, u_x(0) = a,$$
$$u(1) - \beta\, u_x(1) = b,$$

where α, β,a and b are known constants.The function f is a given function of x.

C2: Let $P_n(x)$ denote the Legendre polynomials of degree n, where $x \in [-1,1]$. The Rodriguez formula for $P_n(x)$ is given by

$$P_n(x) = \frac{(-1)^n}{n!\, 2^n} \frac{d^n}{dx^n} (1-x^2)^n$$

for all $n = 0, 1, 2, \cdots$ and $x \in [-1,1]$. First two terms the norm of P_n can be given as

$$P_n(x) = k_n x^n + k_n' x^{n-1} + \cdots$$

$$h_n = \int_{-1}^{1} P_n(x)^2 dx$$

where k_n, k_n' and h_n are given by

$$k_n = \frac{2^n \Gamma(n+1/2)}{n!\, \Gamma(1/2)}, \quad k_n' = 0, \quad h_n = (n+1/2)^{-1}$$

(a) Find the constant α in the following recursion formula

$$(n+1)P_{n+1}(x) = \alpha x P_n(x) - n P_{n-1}(x)$$

(b) Prove that $P_n(x)$ has n real distinct roots in $[-1,1]$.

C3: Homogeneous wave equation with initial and boundary values. Let

$$\begin{aligned} u_{tt} - c^2\, u_{xx} &= 0, \quad (t > 0, \quad 0 < x < L), \\ u(x,0) &= f(x), \quad u_t(x,0) = 0, \quad (0 \le x \le L), \\ u(0,t) &= u(L,t) = 0, \quad (t \ge 0) \end{aligned}$$

where c is an arbitrary constant. Prove the following theorem.

Theorem. *Let $f(x)$ have a continuous fourth derivative for $0 \le x \le L$, and $f(0) = f(L) = 0$, $f''(0) = f''(L) = 0$. Then the initial boundary value problem above has a solution given by*

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\lambda_n c t) \sin(\lambda_n x),$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx,$$

where $\lambda_n = \frac{n\pi}{L}$.

C4: Solve the following integral equation

$$y(x) = f(x) + \lambda \int_0^1 e^{x-s} (1 - 2xs)\, y(s) ds$$

where $f(x)$ is a given function, λ is a constant. Discuss all possibilities.

Section D: Geometry and Topology

D1: Let $f \in k[x_0, \ldots, x_n]$ be an irreducible homogeneous polynomial. Show that $\mathbb{A}^n_k - Z(f)$ is affine, i.e. isomorphic to $\mathbb{A}^n_k$. Here $Z(f)$ denotes the zero set of f.
Hint: First assume that f is linear

D2: Let k be an algebraically closed field. Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n_k$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements. Explain why you need "no nilpotent" elements.

D3: Let X be a set. We can define the cofinite topology on X by declaring a subset $U \subseteq X$ to be open if $X - U$ is finite, or if U is all of X or the empty set. Let Z denote the set of integers with cofinite topology and R denote the set of real numbers with cofinite topology.
(a) Is the topological space R path-connected?
(**b**) Is the topological space R connected?
(c) Is the topological space Z connected?
(d) Is the topological space Z path-connected?

D4: **(a)** Does there exists a finite CW-complex M such that the homology group $H_n(M; \mathbb{Z})$ is infinitely generated for some $n \geq 0$?
(b) Does there exists a connected and compact topological space X such that the homology group $H_n(X; \mathbb{Z})$ is infinitely generated for some $n \geq 0$?