

**BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS**

27 May 2025

Instructions: The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

Time allowed: three hours.

Section A: Algebra

A1: Given a positive integer n , express the group of distance-preserving permutations of $\mathbb{R}^n$ as a semidirect product $U \rtimes V$ where U is the subgroup fixing 0.

A2: Let V be a vector space of countably infinite dimension over a field F . Let $E = \text{End}_F(V)$ and let I be the ideal of E consisting of the endomorphisms with finite rank. Show that the ring E/I has no proper ideals.

A3: Let E be the splitting field for the polynomial $f(X) = X^4 - 12X^2 + 64$ over $\mathbb{Q}$. You may assume that $f(\sqrt{7} + i) = 0$. Find the Galois group $\text{Gal}(E/\mathbb{Q})$ and all the fields F such that $E > F > \mathbb{Q}$.

A4: (a) Let $I = (x^2, y^2)$ in $F[x, y]$, where F is a field of characteristic not equal to 2. Compute the generic initial ideal of I .

(b) Let I be a binomial ideal in a polynomial ring $F[x_1, x_2, \dots, x_n]$ over a field F . Fix any monomial order $<$. Show that I has a Gröbner basis (with respect to $<$) consisting of binomials.

Section B: Analysis

B1: Let $f \in L_2(E) \cap L_4(E)$ with $\|f\|_2 = 16$ and $\|f\|_4 = 2$. Show that $f \in L_3(E)$ with $\|f\|_3 \leq 4$.

B2: Let $T_\alpha : l_2 \longrightarrow l_1 : (x_k)_{k=1}^\infty \mapsto (\alpha_k x_k)_{k=1}^\infty$ for some $\alpha = (\alpha_k)_{k=1}^\infty$. Find all α for which:

- (a) T_α is well-defined.
- (b) T_α is bounded. When it is bounded, find its norm.
- (c) T_α is compact (maps bounded sets in l_2 to precompact sets in l_1).
- (d) T_α is a homeomorphism.

B3: Suppose f is an entire (holomorphic on all of $\mathbb{C}$) function satisfying that $f(z) \in \mathbb{R}$ for all $z \in \mathbb{R}$. Prove that $\overline{f(z)} = f(\bar{z})$ for all $z \in \mathbb{C}$.

B4: Find an explicit conformal (one-to-one, onto, biholomorphic) map of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto the vertical strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$.

Section C: Methods of Applied Mathematics

C1: Let $u_m(x)$, $(m = 0, 1, 2, \dots)$ be one of the classical orthonormal polynomials with weight function $w(x)$ and $x \in [a, b]$. Then prove that the sequence $h_n(x) = w \sum_{k=0}^n u_k(x) u_k(y)$, $(n = 1, 2, \dots)$ where $y \in [a, b]$ defines a distribution over $(-\infty, \infty)$ called the Dirac δ -function $\delta(x - y)$.

C2: Use the singular perturbation method to obtain a uniform approximate solution to the following problem. Assume $0 < \varepsilon \ll 1$ and $0 < t < 1$.

$$\varepsilon y'' + 2y' + e^y = 0, \quad y(0) = y(1) = 0.$$

C3: Homogeneous wave equation with initial and boundary value problem: Let

$$u_{tt} - c^2 u_{xx} = 0,$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0$$

Under what conditions on the data $\{f(x), g(x)\}$ is the above problem well-posed?.

C4: Obtain the most general solution of the integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x + s) y(s) ds$$

when $f(x) = x$ and when $f(x) = 1$. Discuss all cases.

Section D: Geometry, Topology and Number Theory

D1: Let $\alpha = \sqrt[3]{19}$ and $R = \mathbb{A} \cap \mathbb{Q}[\alpha]$. Recall that

$$R = \left\{ \frac{a + b\alpha + c\alpha^2}{3} \text{ with } a \equiv b \equiv c \pmod{3} \right\}.$$

Moreover $3R = P^2Q$ with P and Q distinct prime ideals of R . Prove that the ideal class group is cyclic of order 3 generated by the class containing P .

D2: Chebyshev proved that if $\frac{\Psi(x)}{x}$ tends to a limit as $x \rightarrow \infty$, then this limit equals 1. This exercise outlines another proof based on the identity:

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \frac{\Psi(x)}{x^{s+1}} dt, \quad \sigma \geq 1.$$

(a) Prove that $(1-s)\frac{\zeta'(s)}{\zeta(s)} \rightarrow 1$ as $s \rightarrow 1$.

(b) Let $\beta = \limsup \frac{\Psi(x)}{x}$. Given $\epsilon > 0$, choose $N = N(\epsilon)$ so that $x \geq N$ implies:

$$\Psi(x) \leq (\beta + \epsilon)x.$$

Keeping s real and in the interval $1 < s \leq 2$, split the integral into two parts, $\int_1^N + \int_N^\infty$, and estimate each to obtain the inequality:

$$-\frac{\zeta'(s)}{\zeta(s)} \leq \frac{s(\beta + \epsilon)}{s-1} + C(\epsilon),$$

where $C(\epsilon)$ is a constant independent of s . Use (a) and show that $\beta \geq 1$.

(c) Let $\gamma = \liminf \frac{\Psi(x)}{x}$. Using a similar argument, deduce that $\gamma \leq 1$.

(Thus, if $\frac{\Psi(x)}{x}$ tends to a limit as $x \rightarrow \infty$, then $\gamma = \beta = 1$.)

Some Solutions

A1: [To be supplied.]

A2: [To be supplied.]

A3: [To be supplied.]

B1: We aim to use Hölder's inequality with conjugate exponents p and q (that is $1/p + 1/q = 1$) so that $3 = 2 \cdot 1/p + 4 \cdot 1/q$. Hence $p = q = 2$. Then

$$\|f\|_3^3 = \int_E |f(x)|^3 d\mu = \int_E |f(x)| \cdot |f(x)|^2 d\mu \leq \|f\|_2 \cdot \|f\|_4^2 = 64 = 4^3.$$

Therefore $\|f\|_3 \leq 4$.

B2: (a) Again, by Hölder's inequality, if $\alpha \in l_2$ then $(\alpha_k x_k)_{k=1}^\infty \in l_1$. If $\alpha \notin l_2$ then there exists a sequence $(x_k)_{k=1}^\infty \in l_2$ so that the series $\sum_{k=1}^\infty |\alpha_k x_k|$ diverges. It can be shown directly or by the uniform boundedness principle: if the sequence of functionals $f_N : l_2 \rightarrow \mathbb{R} : (x_k)_{k=1}^\infty \mapsto \sum_{k=1}^N \alpha_k x_k, N \in \mathbb{N}$, is pointwise bounded then it is uniformly bounded. Since $\|f_N\| = \sum_{k=1}^N \alpha_k^2$, we have $\sum_{k=1}^\infty \alpha_k^2 < \infty$. Thus, T_α is well-defined if and only if $\alpha \in l_2$.

(b) The same is the condition for the boundedness of T_α . Since $\|T_\alpha(x)\|_1 \leq \|\alpha\|_2 \cdot \|x\|_2$ and Hölder's inequality is sharp, $\|T_\alpha\| = \|\alpha\|_2$.

(c) Suppose a set A is bounded. Then there is M so that $\|x\|_2 \leq M$ for each $x \in A$. The set $T_\alpha(A)$ is precompact if it is bounded (there is C with $\sum_{k=1}^\infty |\alpha_k x_k| \leq C, x \in A$) and for each ε there is N such that $\sum_{k=N}^\infty |\alpha_k x_k| \leq \varepsilon, x \in A$. Both conditions are valid for $\alpha \in l_2$, as is easy to check. Thus, whenever the operator T_α is well-defined, it is bounded and compact.

(d) There is no α so that T_α is a homeomorphism. Indeed, if T_α is a homeomorphism then its inverse operator is $T_\alpha^{-1} : l_1 \rightarrow l_2 : (y_k)_{k=1}^\infty \mapsto (y_k/\alpha_k)_{k=1}^\infty$. It must be continuous which means $(\sum_{k=1}^\infty |y_k/\alpha_k|^2)^{1/2} \leq C \sum_{k=1}^\infty |y_k|$ for some constant C . Substituting unit vectors e_k yields $|\alpha_k| \geq 1/C$, which contradicts $\alpha \in l_2$.

B3: f has a power series $f(z) = \sum_{n=0}^\infty c_n z^n$ that is convergent for all $z \in \mathbb{C}$. Let $g(z) = \overline{f(\bar{z})}$.

Then $g(z) = \sum_{n=0}^\infty \bar{c}_n z^n$. This is also a power series in z which is convergent for all $z \in \mathbb{C}$

since $|c_n| = |\bar{c}_n|$. Hence g is also entire. If $z \in \mathbb{R}$, then $\bar{z} = z$ and $g(z) = \overline{f(\bar{z})} = f(z)$ by hypothesis. But $\mathbb{R}$ is a subset of $\mathbb{C}$ with an accumulation point. Then by the identity theorem for holomorphic functions, $f(z) = g(z)$ for all $z \in \mathbb{C}$.

B4: Let $f_1(z) = \frac{1+z}{1-z}$; then f_1 maps $\mathbb{D}$ conformally onto the right half plane $P = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Let $f_2(z) = \operatorname{Log} z = \ln z + i \operatorname{Arg} z$, the principal branch of the logarithm with $-\pi < \operatorname{Arg} z < \pi$; then f_2 maps P conformally onto the horizontal strip $Q = \{z \in \mathbb{C} : -\pi/2 < \operatorname{Im} z < \pi/2\}$. Let $f_3(z) = (z - i\pi/2)/\pi$; then f_3 maps Q conformally onto the horizontal strip $R = \{z \in \mathbb{C} : -1 < \operatorname{Im} z < 0\}$. Let $f_4(z) = iz$; then f_4 maps R conformally onto S . The desired conformal map is $f = f_4 \circ f_3 \circ f_2 \circ f_1$.

C4: We use the identity:

$$\sin(x + s) = \sin x \cos s + \cos x \sin s$$

This gives:

$$y(x) = f(x) + \lambda \left(\sin x \int_0^{2\pi} \cos s y(s) ds + \cos x \int_0^{2\pi} \sin s y(s) ds \right)$$

Define the constants:

$$A := \int_0^{2\pi} y(s) \cos s ds, \quad B := \int_0^{2\pi} y(s) \sin s ds$$

So the equation becomes:

$$y(x) = f(x) + \lambda (\sin x \cdot A + \cos x \cdot B)$$

To solve for A and B , multiply both sides by $\cos x$ and integrate:

$$\int_0^{2\pi} y(x) \cos x dx = \int_0^{2\pi} f(x) \cos x dx + \lambda A \int_0^{2\pi} \sin x \cos x dx + \lambda B \int_0^{2\pi} \cos^2 x dx$$

We use:

$$\int_0^{2\pi} \sin x \cos x dx = 0, \quad \int_0^{2\pi} \cos^2 x dx = \pi$$

So:

$$\int_0^{2\pi} y(x) \cos x dx = \int_0^{2\pi} f(x) \cos x dx + \lambda \pi B$$

But the left-hand side is just:

$$\int_0^{2\pi} y(x) \cos x dx = A$$

Thus:

$$A = \int_0^{2\pi} f(x) \cos x dx + \lambda \pi B \quad (1)$$

Similarly, multiply both sides by $\sin x$:

$$\int_0^{2\pi} y(x) \sin x dx = \int_0^{2\pi} f(x) \sin x dx + \lambda A \int_0^{2\pi} \sin^2 x dx + \lambda B \int_0^{2\pi} \sin x \cos x dx$$

Thus:

$$B = \int_0^{2\pi} f(x) \sin x dx + \lambda \pi A \quad (2)$$

Substitute (1) into (2):

$$\begin{aligned} B &= \int f(x) \sin x dx + \lambda \pi \left(\int f(x) \cos x dx + \lambda \pi B \right) \\ \Rightarrow B(1 - \lambda^2 \pi^2) &= \int f(x) \sin x dx + \lambda \pi \int f(x) \cos x dx \end{aligned}$$

Then solve for B and back-substitute for A .

Case 1: $f(x) = 1$.

Then:

$$\int_0^{2\pi} \sin x \, dx = 0, \quad \int_0^{2\pi} \cos x \, dx = 0$$
$$\Rightarrow A = 0, \quad B = 0 \Rightarrow y(x) = 1$$

Case 2: $f(x) = x$.

We compute:

$$\int_0^{2\pi} x \sin x \, dx = -2\pi, \quad \int_0^{2\pi} x \cos x \, dx = 0$$

So:

$$B = \frac{-2\pi}{1 - \lambda^2 \pi^2}, \quad A = \lambda \pi B = \frac{-2\pi^2 \lambda}{1 - \lambda^2 \pi^2}$$

Final solution:

$$y(x) = x + \lambda A \sin x + \lambda B \cos x = x + \lambda \cdot \frac{-2\pi^2 \lambda}{1 - \lambda^2 \pi^2} \sin x + \lambda \cdot \frac{-2\pi}{1 - \lambda^2 \pi^2} \cos x$$

$$\Rightarrow y(x) = x - \frac{2\pi \lambda}{1 - \lambda^2 \pi^2} (\lambda \pi \sin x + \cos x)$$

Final Answers:

- If $f(x) = 1$: $y(x) = 1$

- If $f(x) = x$: $y(x) = x - \frac{2\pi \lambda}{1 - \lambda^2 \pi^2} (\lambda \pi \sin x + \cos x)$