BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

24 May 2022

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3, A4.

Examiner 2: Real Analysis, questions B1, B2.

Examiner 3: Complex Analysis, questions B3, B4.

Examiner 4: Methods of Applied Mathematics, questions C1, C2, C3, C4.

Examiner 5: Geometry, questions D1, D2.

Examiner 6: Topology, questions D3, D4.

Time allowed: three hours.

Section A: Algebra

A1: Let n be a positive integer, p a prime, q a power of p. Recall, $\mathrm{GL}_n(q)$ denotes the group of invertible $n \times n$-matrices over the field with order q. Express, in terms of n and q, the number of Sylow p-subgroups of $\mathrm{GL}_n(q)$. (Hint: Consider the upper triangular matrices.)

A2: You may assume, without proof, the following lemma.

Lemma: *Let L/F be a characteristic 0 field extension with degree 3 and let s and t be elements of $L - F$ such that $s^3 \in F$ and $t^3 \in F$. Then $st \in F$ or $st^2 \in F$.*

Let E be the splitting field over $\mathbb{Q}$ for the polynomial $(X^3 - 2)(X^3 - 3)$.
(a) Determine the Galois group of $E/\mathbb{Q}$ up to isomorphism.
(b) Show that E has a unique subfield K such that K has degree 2 over $\mathbb{Q}$.
(c) Find the number of fields I such that $K \leq I \leq E$.

A3: Let $\mathcal{P}$ and $\mathcal{Q}$ be prime ideals of a unital ring R such that $\mathcal{Q} \subset \mathcal{P}$ (strict inclusion). Show that there exist prime ideals $\mathcal{P}'$ and $\mathcal{Q}'$ such that $\mathcal{Q} \subseteq \mathcal{Q}' \subset \mathcal{P}' \subseteq \mathcal{P}$ and there do not exist any prime ideals $\mathcal{R}$ satisfying $\mathcal{Q}' \subset \mathcal{R} \subset \mathcal{P}'$. (Hint: Choose any $r \in \mathcal{P} - \mathcal{Q}$, then consider some prime ideals that include r and separately consider some prime ideals that do not include r.)

A4: Let F be a field.
(a) Suppose F is algebraically closed. For which positive integers n does there exist an n-dimensional algebra A over F such that, up to isomorphism, A has exactly 3 simple modules and the simple modules have dimensions 1 and 2 and 3?
(b) Give an example to show that the answer to (a) may longer hold if we drop the assumption that F is algebraically closed.

Section B: Analysis

B1: Find the limit $\lim_{n\to\infty} \int_0^n \frac{3\sin\frac{x}{n} + 2\cos\frac{x}{n}}{x^2+1}\, dx$.

B2: Consider the set $A = \{(x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n| < \infty\}$. Find the closure of A in the metric space:
(a) (l_2, d_2),
(b) (A, d_3),
(c) (l_∞, d_∞).
Recall that $d_p(x,y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p}$ for $1 \le p < \infty$, whereas $d_\infty(x,y) = \sup_n |x_n - y_n|$.

B3: Let f be holomorphic in the right half plane $P = \{\, z \in \mathbb{C} : \operatorname{Re} z > 0 \,\}$. Suppose $|f(z)| < 1$ for all $z \in P$ and $f(1) = 0$. Show that $|f'(1)| \le 1/2$, and find all such f for which $|f'(1)| = 1/2$.

B4: Let f be holomorphic in the unit disc $\mathbb{D}$ and suppose $B = \iint_{\mathbb{D}} |f(z)|\, dA(z) < \infty$, where $z = re^{i\theta}$ for $0 \le r < 1$ and $dA(z) = r\, dr\, d\theta$ is the area measure. Fix $a \in \mathbb{D}$ and put $R = 1 - |a|$. Prove that $|f(a)| \le \frac{1}{\pi R^2} B$.

Section C: Methods of Applied Mathematics

C1: Let a second order linear differential equation be given by $Lu = f(x), \quad x \in I = [a, b]$ where L is a second order linear operator and $f(x)$ is continues function in I. Let the boundary conditions be given by $B_1(u) = 0$ at $x = a$ and $B_2(u) = 0$ at $x = b$ where B_1 and B_2 are some (B_1 is not proportional to B_2) first order differential operators.
(a). Find the Green's function of the problem in terms of the solutions of the homogenous problem $Lu = 0$.
(b). Find the solution of the given boundary value problem.
(c). Discuss the existence and uniqueness of the boundary value problem.

C2: Let $u_m(x)$, $(m = 0, 1, 2, \cdots)$ be the classical orthonormal polynomials with weight function $w(x)$ and $x \in [a, b]$. Then prove that the sequence $h_n(x) = w \sum_{k=0}^{n} u_k(x)\, u_k(y)$, $(n = 1, 2, \cdots)$ where $y \in [a, b]$ defines a distribution called the Dirac δ-function $\delta(x - y)$. Prove that $\int_a^b \delta(x - y)\, f(x) dx = f(y)$ if $a < y < b$. It is equal to zero if $b < y < a$.

C3: Homogeneous wave equation with initial and boundary values. Let

$$\begin{aligned} u_{tt} - c^2\, u_{xx} &= 0, \quad (t > 0, \quad 0 < x < L), \\ u(x, 0) &= f(x), \quad u_t(x, 0) = g(x), \quad (0 \le x \le L), \\ u(0, t) &= u(L, t) = 0, \quad (t \ge 0) \end{aligned}$$

Prove the following theorem.

Theorem . *Let $f(x)$ have a continuous fourth derivative and $g(x)$ have a continuous third derivative for $0 \le x \le L$, and $f(0) = f(L) = 0$, $f''(0) = f''(L) = 0$, and $g(0) = g(L) = 0$. Then the initial boundary value problem above has a solution given by*

$$u(x, t) = \sum_{n=1}^{\infty} [A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)] \sin(\lambda_n x),$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx,$$

$$B_n = \frac{2}{\lambda_n c L} \int_0^L g(x) \sin(\frac{n\pi x}{L}) dx.$$

where $\lambda_n = \frac{n\pi}{L}$.

C4: Let $u, v \in C^2(D)$ be twice differentiable functions, where $D \subset R^2$. Let

$$J(u, v) = \int\int_D [(u_x)^2 + (u_t)v + (v_t)^2] dt dx$$

with the initial and boundary conditions (i) $u(0, x) = 2v_t(0, x)$, (ii) $u(t, 0) = 0$, $u(t, \pi) = \varepsilon \sin(\frac{t}{2\varepsilon})$, (iii) $v(0, x) = 0$. Solve this variational problem. Is this problem well posed? Prove your statement (consider the limit $\varepsilon \to 0$).

Section D: Geometry and Topology

D1: Explain in general terms why the coordinate ring of an affine variety is an invariant of the variety and show by an example that the coordinate ring of a projective variety is not such an invariant.

D2: Explain in general terms why we can see an elliptic curve as a projective plane cubic.

D3: First note that a topological group is a group (G, μ) such that G is a Hausdorff space, $\mu : G \times G \to G$ is a continuous function, and the function from G to G which sends g to g^{-1} is also continuous. Let G be a topological group with identity element e and X be a topological space then a continuous function $\alpha : G \times X \to X$ is called an action of G on X if $\alpha(e, x) = x$ and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for every $g, h \in G$ and $x \in X$. An action of G on X is called free if $g \neq e \Rightarrow gx \neq x$ for every $g \in G$ and $x \in X$. An action of G on X is called transitive if for every $x, y \in X$ there exists $g \in G$ such that $gx = y$. Let $\mathbb{S}^1$ denote the unit sphere in $\mathbb{C}$ and $\mathbb{R}^1$ the 1-dimensional Euclidean space. The topological space $\mathbb{S}^1$ can be considered as topological group whose group operation is multiplication of complex numbers and $\mathbb{R}^1$ can be considered as topological group whose group operation is addition of real numbers.
(a) Does there exists an injective continuous function from $\mathbb{S}^1$ to $\mathbb{R}^1$?
(b) Does there exists an surjective continuous function from $\mathbb{S}^1$ to $\mathbb{R}^1$?
(c) Does there exists a free action of $\mathbb{S}^1$ on $\mathbb{R}^1$?
(d) Does there exists a transitive action of $\mathbb{S}^1$ on $\mathbb{R}^1$?
(e) Does there exists a transitive action of $\mathbb{R}^1$ on $\mathbb{S}^1$?
(f) Does there exists a free action of $\mathbb{R}^1$ on $\mathbb{S}^1$?

D4: Let I denote the interval $[0, 1]$ and $\mathbb{S}^1$ denote unit sphere in $\mathbb{C}$. Assume X is the quotient space $\mathbb{S}^1 \times I/\sim$ where we have $(z, t) \sim (w, s)$ if and only if $(s = t)$ and $((z = w)$ or $(z^2 = w^2$ and $t = 0)$ or $(z^2 = w^2$ and $t = 1))$ for every $z, w \in \mathbb{S}^1$ and $s, t \in I$. Compute the homology and cohomology groups $H_i(X; G)$ and $H^i(X; G)$ when $G = \mathbb{Z}$ or $G = \mathbb{Z}/2\mathbb{Z}$.

Some Solutions

A1: Let $V = \mathbb{F}^n$ as a vector space over the field $\mathbb{F}$ with order q. Let $G = \mathrm{GL}_n(q)$ and let S be the p-subgroup of G consisting of those upper triangular matrices whose diagonal entries are 1. The number of ordered $\mathbb{F}$-basis for V is

$$|G| = (q^n - 1)(q^n - q)...(q^n - q^{n-1}) = q^{n(n-1)/2}(q^n - 1)(q^{n-1} - 1)...(q - 1) .$$

Meanwhile, $|S| = q^{n(n-1)/2}$. Evidently, $|G : S|$ is coprime to p, in other words, S is a Sylow p-subgroup of G.

The algebra of upper triangular $n{\times}n$ matrices over $\mathbb{F}$ has an ideal J consisting of the strictly upper triangualr matrices. We have $J = \{s - 1 : s \in S\}$. So conjugation by $N_G(S)$ stabilizes J and the action of $N_G(S)$ on V must fix the chain of subspaces $V > JV > ... > J^{n-1}V > J^nV = 0$. Therefore, $N_G(S)$ is the subgroup of G consisting of those upper triangular matrices whose diagonal entries are non-zero. It follows that $|N_G(S) : S| = (q - 1)^n$. We conclude that the number of Sylow p-subgroups of G is

$$|G : N_G(S)| = \frac{(q^n - 1)(q^{n-1} - 1)...(q^{n-2} - 1)}{(q - 1)^n} .$$

A2: Part (a). Let a and b be the real cube roots of 2 and 3, respectively. Let w be a primitive cube root of unity. Define $A = \mathbb{Q}[a, w]$, which is the splitting field for $X^3 - 2$ over $\mathbb{Q}$.

For a contradiction, suppose $X^3 - 3$ splits over A. Then all the roots to $X^3 - 3$ belong to A and, in particular, $b \in A$. Putting $F = \mathbb{R} \cap A$, then $F = \mathbb{Q}[a] = \mathbb{Q}[b]$. By the lemma, $ab \in \mathbb{Q}$ or $ab^2 \in \mathbb{Q}$. We deduce that 6 or 18 has a cube root in $\mathbb{Q}$, which is a contradiction, as required.

It now follows that the field $E = \mathbb{Q}[a, b, w]$ has degree 3 over $\mathbb{Q}[a, w]$ and hence has degree 18 over $\mathbb{Q}$. Every element of the Galois group $\mathrm{Gal}(E/\mathbb{Q}[w])$ sends a and b to $w^i a$ and $w^j b$ for some i and j. So $\mathrm{Gal}(E/QQ[w])$ is a group with order 9 and exponent 3. hence, $\mathrm{Gal}(E/QQ[w]) \cong C_3{\times}C_3$. By considering group orders, $\mathrm{Gal}(E/\mathbb{Q})$ is a semidirect product having the form $C_2 \ltimes (C_3{\times}C_3)$.

To resolve the action of C_2 on $C_3{\times}C_3$, observe that complex conjugation fixes a and b and acts as $wa \leftrightarrow w^2a$ and $wb \leftrightarrow w^2b$. In conclusion, $\mathrm{Gal}(E/\mathbb{Q}) \cong C_2 \ltimes (C_3{\times}C_3)$ with C_2 acting on $C_3{\times}C_3$ by inversion.

Part (b). The unique such K is $\mathbb{Q}[w]$ by the Fuindamental Theorem of Galois Theory, since $\mathrm{Gal}(E/\mathbb{Q})$ has a unique subgroup with index 2.

Part (c). We have already noted that $\mathrm{Gal}(E/K) \cong C_3{\times}C_3$. There are 1, 4, 1 subgroups of $C_3{\times}C_3$ with orders 1, 3, 9, respectively. Applying the Fundamental Theorem of Galois Theory again, the number of intermediate fields I as specified is $1 + 4 + 1 = 6$.

A3: Let r be as in the hint. Let $\mathfrak{Q}$ denote the set of prime ideals $\mathcal{Q}'$ of R such that $\mathcal{Q} \subseteq \mathcal{Q}' \subset \mathcal{P}$ and $r \notin \mathcal{Q}'$. We partially order $\mathfrak{Q}$ by inclusion. Noting that the unionset of a chain of prime ideals is a prime ideal, we see that every chain in the nonempty poset $\mathfrak{Q}$ has an upper bound. By Zorn's Lemma, $\mathfrak{Q}$ has a maximal element $\mathcal{Q}'$.

Let $\mathfrak{P}$ denote the poset of prime ideals $\mathcal{P}'$ such that $\mathcal{Q}' \subset \mathcal{P}' \subseteq \mathcal{P}$ and $r \in \mathcal{P}$. Noting that the intersectionset of a chain of prime ideals is a prime ideal, another Zorn's Lemma argument shows that $\mathfrak{P}$ has a minimal element $\mathcal{P}'$.

Let $\mathcal{R}$ be a prime ideal such that $\mathcal{Q}' \subseteq \mathcal{R} \subseteq \mathcal{P}'$. If $x \in \mathcal{R}$, then $\mathcal{R} = \mathcal{P}'$, whereas if $x \notin \mathcal{R}$, then $\mathcal{R} = \mathcal{Q}'$.

A4: Part (a). We shall show that such A exists if and only if $n \geq 14$. The hypothesis on F implies that F is the only finite-dimensional division ring over F. Hence, by the Artin–Wedderburn Theorem,

$$A/J(A) \cong \mathrm{Mat}_1(F) \oplus \mathrm{Mat}_2(F) \oplus \mathrm{Mat}_3(F)$$

where $\mathrm{Mat}_k(F)$ denotes the ring of $k\times k$ matrices over F. In particular, $\dim(A) \geq 1^2+2^2+3^2 = 14$. Conversely, write $n = 14+m$ and let B be the unital algebra over F generated by a nilpotent element j such that $j^{m-1} \neq j^m = 0$, with the understanding that $j = 0$ when $m = 0$. We have $B/J(B) \cong F$. Putting

$$A = B \oplus \mathrm{Mat}_2(F) \oplus \mathrm{Mat}_3(F)$$

then A is an n-dimensional algebra over F satisfying the required conditions.

Part (b). Putting $F = \mathbb{R}$ and $A = \mathbb{R}\oplus\mathbb{C}\oplus\mathrm{Mat}_3(\mathbb{R})$, then A satisfies the required conditions, yet $\dim(A) = 1+2+9 = 12$.

B1: Let $f_n := \dfrac{3\sin\frac{x}{n} + 2\cos\frac{x}{n}}{x^2+1} \cdot \chi_{[0,n]}$. Then $f_n \xrightarrow{pw} f$, where $f(x) = \dfrac{2}{x^2+1}$. Also, $|f_n(x)| \leq g(x)$ with $g(x) = \dfrac{5}{x^2+1}$. The function $g(x)$ is integrable on $\mathbb{R}_+$. By LDCT,

$$\lim_{n\to\infty} \int_0^n f_n(x)dx = \int_0^\infty \frac{2\,dx}{x^2+1} = 2\,\arctan x|_0^\infty = \pi.$$

B2: Let $\mathcal{F}in$ be the set of all finite sequences. Clearly, $\mathcal{F}in \subset A \subset c_0$.
a) Since $\mathcal{F}in$ is dense in l_2, we have $\overline{A} = l_2$.
b) Each metric space is closed in itself, so $\overline{A} = A$.
c) Each convergent to zero sequence can be approximated in d_∞ by finite sequences. If $y \in l_\infty \setminus c_0$ then the distance from y to c_0 is positive, as c_0 is closed in l_∞. Hence, $\overline{A} = c_0$.

B3: Let $\varphi(w) = \dfrac{1+w}{1-w}$, which maps the unit disc $\mathbb{D}$ conformally onto P, and let $g = f \circ \varphi$. Then $g : \mathbb{D} \to \mathbb{D}$ and $g(0) = 0$ since $\varphi(0) = 1$. By the Schwarz lemma, $|g'(0)| \leq 1$. But $g'(0) = f'(\varphi(0))\varphi'(0) = 2f'(1)$ since $\varphi'(w) = \dfrac{2}{(1-w)^2}$. Thus $|f'(1)| \leq 1/2$.

The Schwarz lemma also yields that $|g'(0)| = 1$ if and only if g has the form $g(w) = e^{i\theta}w$. So $|f'(1)| = 1/2$ if and only if $f(\varphi(w)) = e^{i\theta}w$. Replace w by $\varphi^{-1}(z) = \dfrac{z-1}{z+1}$. So $|f'(1)| = 1/2$ if and only if $f(z) = e^{i\theta}\,\dfrac{z-1}{z+1}$.

B4: The disc $D(a,R)$ lies in $\mathbb{D}$. Let $0 < r < R$. By the mean value property, $f(a) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$. Then $|f(a)| \leq \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(a+re^{i\theta})|\,d\theta$. Integrating both sides with respect to $r\,dr$ yields

$$\begin{aligned}|f(a)|\,\frac{R^2}{2} = |f(a)|\int_0^R r\,dr &\le \frac{1}{2\pi}\int_0^R\int_0^{2\pi} |f(a+re^{i\theta})|\,d\theta\,r\,dr \\ &= \frac{1}{2\pi}\iint_{D(a,R)} |f(z)|\,dA(z) \le \frac{1}{2\pi}\iint_{\mathbb{D}} |f(z)|\,dA(z) = \frac{1}{2\pi}\,B.\end{aligned}$$

Multiplying through by $\frac{2}{R^2}$ gives the desired result.