BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

28 May 2021

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3.

Examiner 2: Commutative Algebra, question A4.

Examiner 3: Real Analysis, questions B1, B2.

Examiner 4: Methods of Applied Mathematics, questions C1, C2, C3, C4.

Examiner 5: Geometry, questions D1, D2.

Examiner 6: Topology, questions D3, D4.

Time allowed: three hours.

Section A: Algebra

A1: Let G be a finite group, p a prime, H a subgroup of G such that $|H|$ is divisble by p and $|H \cap {}^{g}H|$ is coprime to p for all $g \in G - H$. Show that the index $|G : H|$ is coprime to p.

A2: Let K be the splitting field for $X^3 + 3X^2 + 3$ over $\mathbb{Q}$.

(a) Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ up to isomorphism.

(b) How many fields L are there such that $\mathbb{Q} \leq L \leq K$?

A3: Let F be a field. Let $\mathrm{Mat}_{\mathbb{N}}(F)$ be set of matrices with over F whose rows and columns are indexed by the set of natural numbers. Let R be the ring consisting of those matrices in $\mathrm{Mat}_{\mathbb{N}}(F)$ that have only finitely many entries in each row and each column. Let $F^{\mathbb{N}}$ be the R-module consisting of the coordinate vectors with coordinates indexed by $\mathbb{N}$.

(a) Show that $F^{\mathbb{N}}$ is not a simple R-module.

(b) Explicitly describe a simple R-submodule of $F^{\mathbb{N}}$.

A4: Let F be a field. Suppose G is a Gröbner basis for the non-zero ideal I in $F[x_1, x_2 \ldots, x_n]$ with respect to the lexicographic monomial order with $x_1 > x_2 > \cdots > x_n$. Show that $G \cap F[x_{i+1}, \ldots, x_n]$ is a Gröbner basis for the ideal $I \cap F[x_{i+1}, \ldots, x_n]$.

Section B: Analysis

B1: Suppose a measure μ on $\mathbb{R}$ is such that all subsets of $\mathbb{R}$ are measurable and $\mu(x) > 0$ for each $x \in \mathbb{R}$. For a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, let $\int |f| d\mu < \infty$. Prove that $f(x) = 0$ for all x except maybe a countable set.

B2: Let $T : f(x) \mapsto g(x) = \displaystyle\int_0^x \frac{f(t)dt}{\sqrt{t}}$.

(a) Is T compact as an operator from $C[0,1]$ to $C[0,1]$?

(b) Is T bounded as an operator from $C[0,1]$ to $L_2(0,1)$? If T is bounded, find its norm.

B3: Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc and suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Show that $\dfrac{1 - |f(z)|^2}{1 - |z|^2} \geq \dfrac{1 - |f(0)|}{1 + |f(0)|}$ for all $z \in \mathbb{D}$. (First take care of the case $f(0) = 0$.)

B4: Let f be a nonconstant entire function satisfying the property that for every compact set $K \subset \mathbb{C}$, the preimage set $f^{-1}(K)$ is also compact.

(a) Prove that this property is equivalent to the condition $\lim\limits_{z \to \infty} f(z) = \infty$ which is equivalent to the condition that f has a pole at ∞.

(b) Prove that $f(\mathbb{C}) = \mathbb{C}$, that is, f is onto.

(c) Deduce the fundamental theorem of algebra from above, that is, prove that a nonconstant holomorphic polynomial has a complex root.

Section C: Methods of Applied Mathematics

C1: Let a second order linear differential equation be given by $Lu = f(x), \quad x \in I = [a, b]$ where L is a second order linear operator and $f(x)$ is continues function in I. Let the boundary conditions be given by $B_1(u) = 0$ at $x = a$ and $B_2(u) = 0$ at $x = b$ where B_1 and B_2 are some (B_1 is not proportional to B_2) first order differential operators.

(a) Find the Green's function of the problem in terms of the solutions of the homogenous problem $Lu = 0$.

(b) Find the solution of the given boundary value problem.

(c) Discuss the existence and uniqueness of the boundary value problem.

C2: Use the Poincaré–Linstead method to obtain (first order perturbation) a two term perturbation expansion approximation of the problem $y'' + y = \epsilon[1 - (y')^2]$ with $y(0) = 1$ and $y'(0) = 0$.

C3: Consider

$$\nabla^2 u(r, \theta) = 0, \quad 0 \leq r < a, \tag{1}$$

$$\frac{\partial u(r, \theta)}{\partial r}|_{r=a} + \alpha\, u(a, \theta) = f(\theta), \tag{2}$$

where f is a 2π periodic continuous function and $\alpha > 0$ (a constant).

(a) Solve the above problem formally using separation of variables.

(b) Find reasonable conditions on f so that the formal solution is the solution of the problem.

C4: If ℓ is not preassigned, show that the stationary functions corresponding to the problem $\delta \int_0^\ell [y'^2 + 4(y - 1)]dx = 0$, with $y(0) = 2$ and $y(\ell) = \ell^2$ are of the form $y = x^2 - 2(x/\ell) + 2$. Show that ℓ must be equal to 1.

Section D: Geometry and Topology

D1: Show that there are, up to isomorphism, only two affine quadratic curves but only one projective quadratic curve, where the underlying field is algebraically closed of characteristic different than two. (Here "curve" is taken as smooth and irreducible.)

D2: Show that the complex projective space is quasi-compact both with respect to Euclidean and Zariski topology.

D3: A topological group is a Hausdorff topological space G together with a group structure on G such that the group operation $* : G \times G \to G$ is continuous and the group inversion $(\cdot)^{-1} : G \to G$ is continuous.

(a) Assume that $\mathbb{R}$ denotes the topological group with the group operation $+$ and the topology $\tau = \{ U \subseteq \mathbb{R} \,|\, \forall p \in U, \exists \epsilon \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x - p| < \epsilon \Rightarrow x \in U \}$. Considering the quotient topology on $\mathbb{R}/\mathbb{Q}$ and the quotient group structure on $\mathbb{R}/\mathbb{Q}$. Is $\mathbb{R}/\mathbb{Q}$ a topological group?

(b) If A is a connected subset of a topological space X and $A \subseteq B \subseteq \overline{A}$ then show that B is connected. (Notation: $\overline{A}$ = closure of A in X.)

(c) Let G be a topological group. If G_0 is the component of G that contains the identity element then show that G_0 is a closed normal subgroup of G.

D4: Let X be the CW-complex obtained from $Y = \mathbb{S}^2$ by attaching a cell $\mathbb{D}^3$ by a map of degree 5.

(a) Assume that $\ldots \xrightarrow{\partial_{i+2}} C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} \ldots$ denotes the singular chain complex of X where $0 = C_{-1} = C_{-2} = \ldots$. For an abelian group A and an integer i, let $S_i = \mathrm{Hom}(C_{-i}, A)$, $\widetilde{\partial}_{i+1} = \mathrm{Hom}(\partial_{-i}, 1_A)$ and $K_i = \mathrm{Hom}(A, C_i)$, $\partial'_i = \mathrm{Hom}(1_A, \partial_i)$. Then $\ldots \xrightarrow{\widetilde{\partial}_{i+2}} S_{i+1} \xrightarrow{\widetilde{\partial}_{i+1}} S_i \xrightarrow{\widetilde{\partial}_i} \ldots$ and $\ldots \xrightarrow{\partial'_{i+2}} K_{i+1} \xrightarrow{\partial'_{i+1}} K_i \xrightarrow{\partial'_i} \ldots$ are two chain complexes. Compute the homology groups of these two chain complexes for $A = \mathbb{Z}$, $\mathbb{Z}/5$, $\mathbb{Z}/7$, and $\mathbb{Q}$.

(b) Compute the induced map from $H_i(X;\mathbb{Z})$ to $H_i(X/Y;\mathbb{Z})$ by the quotient map $X \to X/Y$ for i in $\{2, 3\}$.

(c) Compute the induced map from $H^3(X/Y;\mathbb{Z})$ to $H^3(X;\mathbb{Z})$ by the quotient map $X \to X/Y$.

(d) Is the splitting in the universal coefficient theorem for cohomology natural?

Some Solutions

Sol A1: We must show that H contains a Sylow p-subgroup of G. Let T be a Sylow p-subgroup of H. Let S be a Sylow p-subgroup of G containing T. We are to show that $T = S$. Given $g \in N_S(T)$, then $T = T \cap {}^gT \leq H \cap {}^gH$. Since T is non-trivial, $|H \cap {}^gH|$ must be divisible by p. Therefore, $g \in H$. It follows that $g \in T$. So $T = N_S(T)$, in other words, $T = S$, as required.

Sol A2: Part (a). Write $f(X) = X^3 + 3X^2 + 3$. By Eisenstein's Criterion, $f(X)$ is irreducible. So $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to C_3 or S_3. The derivative $3X^2 + 6X$ has roots -2 and 0. We have $f(-2) = 7 > 0 < 3 = f(0)$, so $f(X)$ has exactly one real root. Complex conjugation induces an involution in $\mathrm{Gal}(K/\mathbb{Q})$. Therefore, $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$.

Part (b). The number of intermediate fields L is 6. Indeed, this follows from part (a) and the Fundamental Theorem of Galois Theory, since S_3 has exactly 6 subgroups.

Sol A3: Let S be the R-submodule of $F^{\mathbb{N}}$ consisting of consisting of those elements that have only finitely many non-zero coordinates. The required conclusion of part (a) holds because S is a proper submodule of $F^{\mathbb{N}}$.

For part (b), we shall show that S is simple. Consider a non-zero R-submodule $T \leq S$. We must show that $T = S$. For $i, j \in \mathbb{N}$, let $r_{i,j}$ be the element of R that has (i, j) entry 1 and all other entries 0. Let s_j be the element of S that has j coordinate 1 and all other coordinates 0. The elements s_j comprise an F-basis for S. So there exists $j \in \mathbb{N}$ such that $r_{j,j}T \neq 0$. Hence, $s_j \in T$. For each $i \in \mathbb{N}$, we have $s_i = r_{i,j}s_j \in T$. Therefore $T = S$, as required.

Sol B1: Let $\int |f| d\mu = C$ and $A_n := \{x \in \mathbb{R} : |f(x)| \cdot \mu(x) \geq 1/n\}$ for $n \in \mathbb{N}$. Then the cardinality $\#(A_n)$ of this set does not exceed Cn, so the set A_n is finite. Hence the set $A := \{x : f(x) \neq 0\} = \cup A_n$ is at most countable.

Sol B2: a) The operator $T : C[0, 1] \longrightarrow C[0, 1]$ is compact. Indeed, suppose a set A in $C[0, 1]$ is bounded, so there is C with $||f||_\infty \leq C$ for all $f \in A$. Then, $||Tf||_\infty \leq C \sup_{0 \leq x \leq 1} \int_0^x \frac{dt}{\sqrt{t}} = 2C$. Also, given $\varepsilon > 0$, for $0 \leq x_2 - x_1 \leq \delta$ we have $|Tf(x_2) - Tf(x_1)| \leq \int_{x_1}^{x_2} \frac{|f(t)|dt}{\sqrt{t}} \leq 2C(\sqrt{x_2} - \sqrt{x_1}) \leq 2C\sqrt{\delta} < \varepsilon$ if $\delta < (\varepsilon/2C)^2$. Therefore the set TA is bounded and equicontinuous, so, by Arzela-Ascoli Theorem, it is precompact.

b) The operator $T : C[0, 1] \longrightarrow L_2(0, 1)$ is bounded. Let $||f||_\infty \leq 1$. Then

$$||Tf||_2^2 = \int_0^1 \left(\int_0^x \frac{f(t)dt}{\sqrt{t}} \right)^2 dx \leq \int_0^1 (2\sqrt{x})^2 dx = 2.$$

Hence, $||T|| \leq \sqrt{2}$. If $f \equiv 1$ then $||Tf||_2 = 2$, so $||T|| = \sqrt{2}$.

Sol B3: If $f(0) = 0$, then the inequality to be shown is $1 - |f(z)|^2 \geq 1 - |z|^2$, that is, $|f(z)| \leq |z|$ on $\mathbb{D}$, which immediately follows from the Schwarz lemma.

Otherwise, let $b = f(0) \in \mathbb{D}$ and set $g(z) = \varphi_b(f(z))$, where $\varphi_b(w) = \dfrac{b - w}{1 - \bar{b}\,w}$ is the holomorphic automorphism of $\mathbb{D}$ that exchanges 0 and b. Then $g : \mathbb{D} \to \mathbb{D}$ is holomorphic,

$g(z) = \dfrac{b - f(z)}{1 - \overline{b}\, f(z)}$, $g(0) = 0$, and hence $|g(z)| \le |z|$ on $\mathbb{D}$. So

$$|z|^2 \ge |g(z)|^2 = \frac{b - f(z)}{1 - \overline{b}\, f(z)}\, \frac{\overline{b} - \overline{f(z)}}{1 - b\, \overline{f(z)}} = \frac{|b|^2 - b\, \overline{f(z)} - \overline{b}\, f(z) + |f(z)|^2}{1 - b\, \overline{f(z)} - \overline{b}\, f(z) + |b|^2 |f(z)|^2}\,.$$

Then $1 - |z|^2 \le \dfrac{1 + |b|^2|f(z)|^2 - |b|^2 - |f(z)|^2}{1 - b\, \overline{f(z)} - \overline{b}\, f(z) + |b|^2|f(z)|^2} = \dfrac{(1 - |b|^2)(1 - |f(z)|^2)}{|1 - \overline{b}\, f(z)|^2}$. Thus

$$\frac{1 - |f(z)|^2}{1 - |z|^2} \ge \frac{|1 - \overline{b}\, f(z)|^2}{1 - |b|^2} \ge \frac{(1 - |b|\, |f(z)|)^2}{(1 - |b|)(1 + |b|)} \ge \frac{(1 - |b|)^2}{(1 - |b|)(1 + |b|)} = \frac{1 - |f(0)|}{1 + |f(0)|}\,.$$

Sol B4: (a) Suppose the property holds and let $M > 0$. Denote by $D(a, r)$ the open disc centered at a with radius r. Then $f^{-1}(\overline{D(0, M)})$ is compact and hence lies in $\overline{D(0, R)}$ for some $R > 0$, that is, $\overline{D(0, M)} \subset f(\overline{D(0, R)})$. So if $|z| > R$, then $|f(z)| > M$. This shows $\lim_{z\to\infty} f(z) = \infty$. Conversely, if $\lim_{z\to\infty} f(z) = \infty$ and $K \subset \mathbb{C}$ is compact, then $K \subset \overline{D(0, M)}$ for some $M > 0$. There is $R > 0$ such that if $|z| > R$, then $|f(z)| > M$, that is, $f^{-1}(\overline{D(0, M)}) \subset \overline{D(0, R)}$. Then $f^{-1}(K)$ lies in $\overline{D(0, R)}$, hence it is bounded, and since it is closed by the continuity of f, it is compact. (We have not used the analyticity of f yet.)

Let $g(z) = f(1/z)$. Then $\lim_{z\to 0} g(z) = \lim_{w\to\infty} f(w)$. This limit is ∞ if and only if g has a pole at 0 if and only if f has a pole at ∞.

(b) Suppose there is a $c \in \mathbb{C}$ such that $f(z) \ne c$ for all $z \in \mathbb{C}$. Let $h(z) = 1/(f(z) - c)$; then h is also entire. But $\lim_{z\to\infty} h(z) = 0$ since by above $\lim_{z\to\infty} f(z) = \infty$. This shows h is bounded and by the Liouville theorem, it is constant. Then f is also constant contrary to hypothesis. Thus there is no such c.

(c) Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with $n \ge 1$ and $q(z) = p(1/z)$. Then $q(z) = \dfrac{a_n}{z^n} + \dfrac{a_{n-1}}{z^{n-1}} + \cdots + \dfrac{a_1}{z} + a_0$ and $\lim_{z\to 0} q(z) = \infty$. By above, p has a pole at ∞ and hence is onto. Consequently, p takes the value 0 at some $a \in \mathbb{C}$ and thus has a as a root.