BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

12 January 2021

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3.

Examiner 2: Commutative Algebra, question A4.

Examiner 3: Real Analysis, questions B1, B2.

Examiner 4: Methods of Applied Mathematics, questions C1, C2, C3, C4.

Examiner 5: Topology, questions D1, D2.

Time allowed: three hours.

Section A: Algebra

A1: The orthogonal group $O(2)$ is the group consisting of the rotations and reflections fixing a distinuished point of the Euclidian plane. You may assume all observations implicit in that definition of $O(2)$. Determine, up to isomorphism, all the finite subgroups of $O(2)$.

A2: Let $K = \mathbb{Q}\left[\sqrt{2}, \sqrt{3}, \sqrt{(2+\sqrt{2})(3+\sqrt{3})}\right]$.

(a) Let α be a root to $X^2 - (2+\sqrt{2})(3+\sqrt{3})$. Let $g \in \mathrm{Aut}(K)$ such that $g(\sqrt{2}) = -\sqrt{2}$ and $g(\sqrt{3}) = \sqrt{3}$. By calculating $g(\alpha^2)/\alpha^2$, show that $g(\alpha) = \pm(1-\sqrt{2})\alpha$.

(b) Prove a similar equality for $h(\alpha)$, where $h \in \mathrm{Aut}(K)$ with $h(\sqrt{2}) = \sqrt{2}$ and $h(\sqrt{3}) = -\sqrt{3}$.

(c) Show that $g^2(\alpha) = h^2(\alpha) = -\alpha$.

(d) Show that g and h do not commute.

(e) Determine the Galois group $\mathrm{Aut}(K) = \mathrm{Gal}(K/\mathbb{Q})$ up to isomorphism.

(f) What is the number of subfields L of K such that L is not a Galois extension of $\mathbb{Q}$?

A3: **(a)** Classify, up to isomorphism, the finite-dimensional semisimple algebras A over $\mathbb{R}$ such that $Z(A) \cong \mathbb{R}$.

(b) Give an example of a non-semisimple finite-dimensional algebra B over $\mathbb{R}$ such that $Z(B) \cong \mathbb{R}$.

A4: Show that a squarefree monomial ideal is an intersection of monomial prime ideals.

Section B: Analysis

B1: Prove or disprove by means of an appropriate counterexample

(a) For each $f \in L_1(\mathbb{R})$ we have $\int_{-t}^{t} f(x)dx \to \int_{-\infty}^{\infty} f(x)dx$ as $t \to \infty$.

(b) If $f \in L_p(\mathbb{R})$ for each p with $1 \le p < \infty$ then $f \in L_\infty(\mathbb{R})$.

B2: **(a)** Let $\mathcal{F}in$ be the set of all finite sequences ($x = (x_k)_{k=1}^{\infty} \in \mathcal{F}in$ if it has a finite number of nonzero terms) and $d(x,y) = \sup_{k=1} |x_k - y_k|$. Find a completion of the metric space $(\mathcal{F}in, d)$.

(b) Let $\mathcal{P}$ be the space of all polynomials. For $P(x) = \sum_{k=0}^{N} a_k x^k, Q(x) = \sum_{k=0}^{M} b_k x^k$, define $d(P,Q) = \sup_k |a_k - b_k|$. Find a completion of $(\mathcal{P}, d)$.

B3: Suppose p is a holomorphic polynomial of degree n that satisfies $|p(z)| \le 1$ for all $z \in \mathbb{C}$ with $|z| < 1$. Prove that $|p(z)| \le |z|^n$ for all $z \in \mathbb{C}$ with $|z| \ge 1$.

B4: Let $K \subset \mathbb{C}$ be compact, and for $z \in \mathbb{C}$, let $dA(z) = r\,dr\,d\theta$ be the area measure on $\mathbb{C}$.

(a) For $w \in \mathbb{C}$, define $f(w) = \displaystyle\int_K \frac{dA(z)}{|z-w|}$. Show that f is bounded on $\mathbb{C}$.

(b) For $w \in \mathbb{C}$, define also $g(w) = \displaystyle\int_K \frac{dA(z)}{z-w}$. Show that g is holomorphic on $\mathbb{C} \setminus K$.

Section C: Methods of Applied Mathematics

C1: Let $L(u) \equiv a(x)u'' + b(x)u' + c(x)u = f(x)$ be a linear second order differential equation. Here $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are some given continuous functions in an interval I. Let $u_1(x)$ be a solution of the associated homogenous equation. Find the second solution $u_2(x)$ of the associated homogenous equation and the particular solution $u_p(x)$.

C2: Solve the following boundary value problem by using the Green's Function method.

$$\begin{aligned} &u_{xx} = f(x), \quad x \in (0,1), \\ &u(0) - \alpha\, u'(0) = 0,, \\ &u(1) - \beta\, u'(1) = 0,, \end{aligned}$$

where α and β are constants and $f(x)$ is any continuous function in $[0,1]$.

C3: Let a functional J be given by $J[y] = \int_0^b (y')^2\, dx$ where the upper end point is moving on the curve $g(x) = \sin x$ and $y(0) = 2$. Show that the critical points of the functional J are of the form $y(x) = 2 + 2\,x\, \cos b$ where b satisfies the transcendental equation $2 + 2b \cos b - \sin b = 0$. Also show that the smallest possible value of b is between $\pi/2$ and $3\pi/4$.

C4: Let a boundary and initial value problem be given by

$$\begin{aligned} &u_{tt} + k u_t = c^2 u_{xx} + F(x,t), \quad 0 < x < L,\ t > 0 \\ &u(x,0) = f(x), \quad u_t(x,t) = g(x), \quad 0 < x < L,\ t \geq 0, \\ &u(0,t) = r(t), \quad u(L,t) = s(t), \quad t \geq 0 \end{aligned}$$

where c and $k > 0$ are constants and $F(x,t)$, $f(x)$, $g(x)$, $r(t)$ and $s(t)$ are given functions (data of the initial and boundary value problem). Assuming the existence of the solution of this initial and boundary value problem prove that there exists at most one solution . [**Hint**: If you wish you may use the energy functional $E(t) = \frac{1}{2} \int_0^L \left((u_t)^2 + c^2\, (u_x)^2 \right)\, dx$.]

Section D: Geometry and Topology

D1: Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements. Show in particular how the "nilpotent" concept comes into play.

D2: Let Q_1 and Q_2 be the quadric surfaces in $\mathbb{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic and a line.

D3: First note that a topological group is a group (G, μ) such that G is a Hausdorff space, $\mu : G \times G \to G$ is a continuous function, and the function from G to G which sends g to g^{-1} is also continuous. Now assume that (G, μ) is a topological group and e denotes the identity element of G. Then prove the following:

(a) If U is an open subset of G and $e \in U$ then there exists an open subset V of G such that $e \in V$, $V \subseteq U$, and $V = V^{-1}$ where V^{-1} denotes the set $\{ g^{-1} \mid g \in V \}$.

(b) If U is an open subset of G, $e \in U$, and m is a positive integer then there exists an open subset V of G such that $e \in V$ and $V^m \subseteq U$ where $V^1 = V$ and V^n denotes the set $\{ \mu(g_1, g_2) \mid g_1 \in V^{n-1} \text{ and } g_2 \in V \}$ for $n \geq 2$.

(c) If $x \in G$, K is closed subsed of G, and $x \notin K$ then there exists two open subsets W and Z of G such that $x \in W$, $K \subseteq Z$, and $W \cap Z = \emptyset$.

D4: Let A be a subcomplex of a finite simplicial complex X and i denote the inclusion map of A into X. Assume that $|A|$ is homemorphic to $\mathbb{S}^1$ and there exists a group homomorphism $\rho : H_1(|X|; \mathbb{Z}) \to H_1(|A|; \mathbb{Z})$ such that $\rho \circ i_*$ is the identity morphism on $H_1(|A|; \mathbb{Z})$. Show that there exists a continuous function r from $|X|$ to $|A|$ such that $r \circ i$ is the identity function on $|A|$ up to homotopy.

Some Solutions

A1: Up to isomorphism, the finite subgroups of $O(2)$ are the cyclic groups C_n and the dihedral groups D_{2n}, where n is any positive integer. By considering the rotational and Euclidian symmetries of a regular n-gon, suitably interpreted in the cases $n \in \{1, 2\}$, we see that all of those groups do occur.

Conversely, let G be a finite subgroup of $O(2)$. Let C be the set of rotations in G. Since G is finite, there exists a positive integer n such that every element of C is a rotation though an angle of $2\pi u/n$ for some integer u. So C is a subgroup of the cyclic group generated by a rotation through $2\pi/n$. Every subgroup of a cyclic group is cyclic. So C is cyclic, say, with generator g.

If $G = C$ then we are finished. Suppose $G > C$ and let $h \in G - C$. Then h and hg are reflections, so $h = h^{-1}$ and $hg = (hg)^{-1} = g^{-1}h^{-1} = g^{-1}h$. Therefore $hgh^{-1} = g^{-1}$ and G is a dihedral group.

A2: Part (a). We have $\dfrac{g(\alpha^2)}{\alpha^2} = \dfrac{2-\sqrt{2}}{2+\sqrt{2}} = (2-\sqrt{2})^2/2 = 3 - 2\sqrt{2} = (1-\sqrt{2})^2$.

Part (b). Let $\lambda = \dfrac{1-\sqrt{3}}{\sqrt{2}}$. Then $\dfrac{h(\alpha^2)}{\alpha^2} = \dfrac{3-\sqrt{3}}{3+\sqrt{3}} = (3-\sqrt{3})^2/6 = 2-\sqrt{3} = \lambda^2$. Hence $h(\alpha) = \pm\lambda\alpha$.

Part (c). We have $g(\alpha) = (1+\sqrt{2})(1-\sqrt{2})\alpha = -\alpha = (1+\sqrt{3})(1-\sqrt{3})/2 = h^2(\alpha)$.

Part (d). Replacing g or h with g^3 or h^3 if necessary, we may assume that $g(\alpha) = (1-\sqrt{2})\alpha$ and $h(\alpha) = \lambda\alpha$. Noting that $g(\lambda) = -\lambda$, we have $hg(\alpha) = \lambda(1-\sqrt{2})\alpha = -gh(\alpha)$.

Part (e). We have $|\mathrm{Aut}(K)| = |K : \mathbb{Q}| = 8$. By part (c), g and h have order 4. In view of part (d), $\mathrm{Aut}(K) \cong Q_8$.

Part (f). Every subgroup of Q_8 is normal. So by the Fundamental Theorem of Galois Theory, every subfield of K is a Galois extension of $\mathbb{Q}$. The number of such L is 0.

A3: Part (a). For a positive integer n and a division ring Δ, let $\mathrm{Mat}_n(\Delta)$ denote the ring of $n \times n$ matrices over Δ. The algebras satisfying the condition on (a) are precisely those that are isomorphic to $\mathrm{Mat}_n(\mathbb{R})$ or $\mathrm{Mat}_n(\mathbb{H})$ for some n. Plainly, those algebras do satisfy the condition.

Conversely, given A, then $A \cong \bigoplus_i A_i$ as a finite direct sum of simple algebras A_i. Since $Z(A) \cong \bigoplus_i Z(A_i)$, the direct sum has only one term, and A is simple, in other words, $A \cong \mathrm{Mat}_n(\Delta)$ for some n and some division algebra Δ over $\mathbb{R}$. We have $Z(A) \cong Z(\Delta)$. The only division algebras over $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$ and $\mathbb{H}$, which have centres $\mathbb{R}$ and $\mathbb{C}$ and $\mathbb{R}$, respectively.

Part (b). We can take B to be the ring of upper triangular $m \times m$ matrices over $\mathbb{R}$, where m is any integer with $m \geq 2$.

B1: (a) True: Let $g_t(x) = f(x) \cdot \chi_{[-t,t]}$. Then $|g| \leq |f| \in L_1$ and $g \xrightarrow{pw} f$. By LDCT, $g \to f$ in L_1.

(b) False: Take $f(x) = -\log x$ for $0 < x < 1$ and $f = 0$ otherwise. Then $\int_0^1 f^p dx < \infty$, since $f^p(x) < 1/\sqrt{x}$ near 0, but essential supremum of f is ∞.

B2: Clearly, $\widetilde{(\mathcal{F}in, d)} = c_0$, hence $\widetilde{(\mathcal{P}, d)} = \{f(x) = \sum_{k=0}^{\infty} a_k x^k : a_k \to 0\}$.

B3: Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$. By continuity also $|p(z)| \le 1$ also when $|z| = 1$. By the Cauchy estimates, $|p^{(n)}(0)| \le \dfrac{1n!}{1^n}$, which gives $|a_n| \le 1$. The function

$$q(z) = \frac{p(z)}{z^n} = \frac{a_0}{z^n} + \cdots + \frac{a_{n-1}}{z} + a_n$$

is holomorphic for $|z| > 1$. But $\lim\limits_{z\to\infty} |q(z)| = |a_n| \le 1$ and $|q(z)| = \dfrac{|p(z)|}{|z|^n} \le 1$ for $|z| = 1$. Then by the maximum modulus theorem, $|q(z)| \le 1$ for $|z| \ge 1$, and this is the desired result.

B4: (a) Fix an open disc $D(0,R)$ that contains K. There is a $d > 0$ such that $|z - w| \ge d$ for all $w \notin D(0,R)$, because the distance between two disjoint compact sets (K and $\partial D(0,R)$) is positive. For such w, $f(w) \le A(K)/d$.

Next let $w \in D(0,R)$. Make the change of variables $\zeta = z - w$. Let $C_w = K - w$; then C_w is also compact and $C_w \subset D(0,2R)$ for any such w. Since the area (Lebesgue) measure is translation-invariant, for such w,

$$f(w) = \int_C \frac{dA(\zeta)}{|\zeta|} \le \int_{D(0,2R)} \frac{dA(\zeta)}{|\zeta|}.$$

Now by using polar coordinates,

$$f(w) \le \int_0^{2\pi}\int_0^{2R} \frac{r\,dr\,d\theta}{r} = 4\pi R$$

for such w. Setting $M = \max\{A(K)/d, 4\pi R\}$, for all $w \in \mathbb{C}$, $f(w) \le M$.

(b) First solution: Let Δ be a triangle in $\mathbb{C} \setminus K$. By the Fubini theorem and since $z \notin \Delta$,

$$\int_{\partial\Delta} g(w)\,dw = \int_{\partial\Delta}\int_K \frac{dA(z)}{z-w}\,dw = \int_K\int_{\partial\Delta} \frac{dw}{z-w}\,dA(z) = \int_K 0\,dA(z) = 0.$$

Then by the Morera theorem, f is holomorphic on $\mathbb{C} \setminus K$. The use of the Fubini theorem is justified because $\displaystyle\int_{\partial\Delta}\int_K \frac{dA(z)}{|z-w|}\,|dw| \le \int_{\partial\Delta} M\,|dw| = M\,\text{Length}(\partial\Delta) < \infty$ by part **(a)**.

Second solution: For $a \in \mathbb{C} \setminus K$, let $r > 0$ be such that $D(a,r) \subset \mathbb{C} \setminus K$. For $w \in D(a,r)$ and $z \in K$, $\left|\dfrac{w-a}{z-a}\right| \le \dfrac{|w-a|}{r} = s < 1$. Then for such z and w,

$$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = \frac{1}{(z-a)\left(1 - \dfrac{w-a}{z-a}\right)} = \sum_{m=0}^{\infty} \frac{(w-a)^m}{(z-a)^{m+1}}$$

and the series converges uniformly for $z \in K$ since s is independent of $z \in K$. Then we can exchange the order of integration and summation in the definition of g to obtain

$$g(w) = \int_K \sum_{m=0}^{\infty} \frac{(w-a)^m}{(z-a)^{m+1}}\,dA(z) = \sum_{m=0}^{\infty} (w-a)^m \int_K \frac{dA(z)}{(z-a)^{m+1}} = \sum_{m=0}^{\infty} c_m\,(w-a)^m$$

for all $w \in D(a,r)$, where $c_m = \displaystyle\int_K \frac{dA(z)}{(z-a)^{m+1}}$. Since g has a power series representation around every point $a \in \mathbb{C} \setminus K$, it is holomorphic on this set. (The convergence of the power series in $D(a,r)$ follows from the proof; in fact, $|c_m| \le A(K)/r^{m+1}$.)

Third solution: By Theorem 10.7 of W. Rudin, *Real and Complex Analysis*, g is holomorphic on $\mathbb{C} \setminus K$. In fact, the second solution is just the proof of this theorem in the special case of the question.

Second and third solutions are independent of part **(a)**.

D1: Assume first that B is isomorphic to the affine coordinate ring of some algebraic set Y in $\mathbb{A}^n$, i.e. $B \cong k[X_1, \ldots, X_n]/I(Y)$. Let x_i be the images of X_i in B. Then B is finitely generated by $x_1, \ldots, x_n$. Since $I(Y)$ is a radical ideal, B has no non-zero nilpotent elements. Conversely assume that B is finitely generated by $b_1, \ldots, b_n$ and that it has no nilpotent elements. Consider the ring morphism $\phi : k[X_1, \ldots, X_n] \to B$ given by $\phi(X_i) = b_i$. The map is surjective and we have $B \cong k[X_1, \ldots, X_n]/\ker \phi$. Since B has no nilpotent elements, $\ker \phi$ is a radical ideal and by Hilbert's Nullstellensatz $I(Z(\ker \phi)) = \ker \phi$. Hence by definition B is isomorphic to the affine coordinate ring of $Z(\ker \phi)$.

D2: Let $[w : x : y : z]$ be the homogeneous coordinates in $\mathbb{P}^3$.

If $x = 0$ then either $y = 0$ or $w = 0$. In the subcase $y = 0$, we get the points $[1 : 0 : 0 : 0]$ and $[0 : 0 : 0 : 1]$. In the subcase $w = 0$, we get the line $[0 : 0 : y : z]$ where $[y : z] \in \mathbb{P}^1$.

If $x \neq 0$, then $wyz \neq 0$. Without loss of generality we assume that $w = 1$ and get the solution space $[1 : x : x^2 : x^3]$ where $x \in k$, which is the twisted cubic in $\mathbb{P}^3$.