BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

24 October 2018

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3.

Examiner 2: Commutative Algebra, question A4.

Examiner 3: Real Analysis, questions B1, B2.

Examiner 4: Methods of Applied Mathematics, questions C1, C2, C3, C4.

Examiner 5: Topology, questions D1, D2.

Time allowed: three hours.

Section A: Algebra

A1: State, without proof, which finite groups can appear as the Galois group of (the splitting field of) an irreducible quartic equation over $\mathbb{Q}$. Choose one of those groups, and exhibit an irreducible quartic over $\mathbb{Q}$ with that Galois group.

A2: Find, up to conjugacy, all the subgroups of S_5. Give a good proof (systematic and easy to read) that your list is complete.

A3: Classify the simple $\mathbb{Q}G$-modules when (a) $G = C_3$, (b) $G = S_3$, (c) $G = Q_8$.

A4: Let I be a homogeneous ideal in a polynomial ring $F[x_1, x_2, \ldots, x_n]$, where F is a field. Let $<_1$ and $<_2$ be two monomial orders on the set of monomials. Show that we cannot have $LT_{<_1}(I) \subsetneq LT_{<_2}(I)$, (that is, a lead term ideal cannot be properly contained in another lead term ideal).

Section B: Analysis

B1: Let E be a subset of $\mathbb{R}$ with positive Lebesgue measure. Prove or disprove by means of an appropriate counterexample
a) $\exists\, a \neq b \in E: \ a - b \in \mathbb{Q}$,
b) $\exists\, a, b \in E: \ a - b \notin \mathbb{Q}$.

B2: Let $X = C[0,1], \alpha \in X$ and $T : X \longrightarrow X : f \mapsto \alpha \cdot f$. In terms of α, give a condition when the operator T is compact.

Section C: Methods of Applied Mathematics

C1: Consider the following sequence:

$$h_n(x) = \begin{cases} 0 & \text{if } x \leq -1/n \\ (nx+1)/2 & \text{if } -1/n \leq x \leq 1/n \\ 1 & \text{if } x \geq 1/n \end{cases}$$

(i) Prove that $h_n(x) \to \theta(x)$ where $\theta(x)$ is the step function and
(ii) $dh_n(x)/dx \to \delta(x)$. Hence formally we may write that $d\theta(x)/dx = \delta(x)$

C2: Let a second order linear differential equation be given by $Lu = f(x),\ \ x \in I = [a, b]$ where L is a second order linear operator and $f(x)$ is continuous function in I. Let the boundary conditions be given by $B_1(u) = 0$ at $x = a$ and $B_2(u) = 0$ at $x = b$ where B_1 and B_2 are some (B_1 is not proportional to B_2) first order differential operators.
(a) Find the Green's function of the problem in terms of the solutions of the homogenous problem $Lu = 0$.
(b) Find the solution of the given boundary value problem.
(c) Discuss the existence and uniqueness of the boundary value problem.

C3: Let u be a function of $t \in I$ satisfying the second order equation

$$\ddot{u} + \dot{u} - u + u^3 = 0$$

which is called the Duffing equation. Discuss the stability of the equilibrium solutions of this equation.

C4: Let $u, v \in C^2(D)$ be twice differentiable functions, where $D \subset R^2$. Let

$$J(u, v) = \int\int_D [(u_x)^2 + (u_t)v + (v_t)^2] dtdx$$

with the initial and boundary conditions

(i) $u(0, x) = 2v_t(0, x)$, (ii) $u(t, 0) = 0,\ u(t, \pi) = \varepsilon \sin(t/2\varepsilon)$, (iii) $v(0, x) = 0$.

Solve this variational problem. Is this problem well posed? Prove your statement (consider the limit $\varepsilon \to 0$).

Section D: Geometry and Topology

D1: Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology with basis given by products $U \times V$, where $U \subset X$ and $V \subset Y$ are open. Show that the following hold.

(a) The projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous, and the product topology is the smallest topology for which this is true.

(b) If X is compact then the projection $\pi_Y : X \times Y \to Y$ is closed. (Note that a map $f : X \to Y$ is said to be closed if $f(C)$ is closed in Y for all closed $C \subset X$.)

D2: (a) Consider the space X which is the union of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^2$ and the line segment between the north and south poles. Give it a CW-complex structure and compute its homology directly.

(b) Show that the space X in part (a) is homotopy equivalent to the one-point union $\mathbb{S}^2 \vee \mathbb{S}^1$ of a 2-sphere and a circle. Use this to give an easier computation of $H_*(X)$.