BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

30 May 2016

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3.

Examiner 2: Commutative Algebra, question A4.

Examiner 3: Real Analysis, questions B1, B2.

Examiner 4: Complex Analysis, questions B3, B4.

Examiner 5: Methods of Applied Mathematics, questions C1, C2, C3, C4.

Examiner 6: Geometry, questions D1, D2.

Examiner 7: Topology, questions D3, D4.

Time allowed: three hours.

Section A: Algebra

A1: Given a field K of characteristic zero, let $G_K = \mathrm{Gal}(E/K)$ where E is the splitting field for $(X^2-2)(X^3-3)$ over K. Find, up to isomorphism, all the possible values of the Galois group G_K. For each possible value of G_K, give an example of such a field K.

A2: Let F be a field, let V be a vector space over F, let I be a linearly independent set in V and let S be a spanning set in V such that $I \subseteq S$. Prove that V has a basis B satisfying $I \subseteq B \subseteq S$. (Warning: V is not necessarily finite-dimensional.)

A3: Let G be the group of rigid symmetries of a regular dodecahedron (the group of distance-preserving permutations of $\mathbb{R}^3$ that stabilize the set of vertices of the dodecahedron).

(a) Show that $|G| = 120$.

(b) Determine G up to isomorphism, and find the character table of G. (Warning: G is not isomorphic to S_5.)

A4: Let F be a field and let R be the polynomial ring in $n$variables. Let I be an ideal in R. Show that I has only finitely many distinct initial ideals, in other words, show that there are only finitely many ideals having the form $\mathrm{in}_<(I)$ as $<$ runs over the monomial orders on R.

Section B: Analysis

B1: (a) Let $f \in L_2(\mathbb{R})$. Prove that $\int_{(n,\infty)} f^2\,dx \longrightarrow 0$ as $n \longrightarrow \infty$.

(b) Let $f \in L_1(\mathbb{R})$. Is it true that $\int_{(n,\infty)} f^2\,dx \longrightarrow 0$ as $n \longrightarrow \infty$?

B2: Find the norm of the functional $f(x) = \int_0^\pi x(t)\cos t\,dt$ in each of the spaces:

(a) $L_1(0,\pi)$, **(b)** $L_2(0,\pi)$, **(c)** $C[0,\pi]$.

B3: Let f be holomorphic in the unit disc $\mathbb{D}$. Prove that there is a sequence $\{z_n\}$ in $\mathbb{D}$ such that $|z_n| \to 1$ and $\{f(z_n)\}$ is bounded.

B4: Suppose that h is holomorphic in an open set containing the closed unit disc $\overline{\mathbb{D}}$, has N zeros in $\mathbb{D}$ counted with multiplicity, and has no zero on the unit circle $\partial\mathbb{D}$. Prove that $\mathrm{Re}(h)$ (the real part of h) has at least $2N$ zeros on $\partial\mathbb{D}$.

Section C: Methods of Applied Mathematics

C1: Prove that $\displaystyle\int_{-\infty}^{\infty} \delta(x^2 - a^2)dx = \frac{1}{2|a|}[\delta(x-a) + \delta(x+a)]$.

C2: Let $y = f(x)$ be the graph of a smooth function f where $x \in [0,1]$. We define four functionals which all have geometrical meaning:

(i) The length of the curve over $[0,1]$ is $\ell = \displaystyle\int_0^1 \sqrt{1+(y')^2}\,dx$.

(ii) The area of the region under the curve over $[0,1]$ is $a = \displaystyle\int_0^1 y\,dx$.

(iii) When the region under the curve over $[0,1]$ is rotated about the x-axis, the volume of the solid is $v = \pi^2 \displaystyle\int_0^1 y^2\,dx$.

(iv) When the region under the curve over $[0,1]$ is rotated about the x-axis, the area of the surface traversed by the curve is $s = 2\pi \displaystyle\int_0^1 y\sqrt{1+(y')^2}\,dx$.

Assuming $y(0) = y(1) = 0$,

(a) find the shortest length when the area is fixed,

(b) find the maximum surface area when the volume is fixed.

C3: Let $y(x) = f(x) + \lambda \displaystyle\int_0^1 (x-s)\,y(s)\,ds$ where $f(x) = x^a$ with $a \in \mathbb{R}$. Find the values of λ and a such that:

(a) There is a unique solution $y(x)$. Give the solution.

(b) There are infinitely many solutions $y(x)$. Give these solutions.

(c) There are no solutions $y(x)$.

C4: Solve the following initial value problems.

(a) $u_{xx} - 3u_{xy} + 2u_{yy} = 0$ with $u(x,x) = f(x)$ and $u_y(x,x) = g(x)$.

(b) $u_{xx} - 3u_{xy} + 2u_{yy} = 0$ with $u(x,-x) = f(x)$ and $u_y(x,-x) = g(x)$.

Section D: Geometry and Topology

D1: Let X and Y be two curves in $\mathbb{P}^2_k$ over an algebraically closed field k. Show that X and Y always intersect non-trivially. Explain, in general terms, where you need the algebraic closure of k for this result. Give an example of two plane curves which do not intersect in $\mathbb{P}^2_{\mathbb{R}}$.

D2: Let H_i and H_j be hyperplanes in $\mathbb{P}^n_k$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Take k as algebraically closed. Show that any regular function on $\mathbb{P}^n - (H_i \cap H_j)$ is constant.
(Hartshorne Ex. I.3.8.)

D3: Let X and Y be topological spaces.

(a) Let $f : X \to Y$ be a continuous map. Prove that if X is compact then the image of f is compact.

(b) Let $f, g : X \to Y$ be continuous maps. Prove that if Y is Hausdorff then $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.

D4: Let X be a simplicial complex. Let A and B be subcomplexes of X such that $A \cup B = X$.

(a) Show that the (simplicial) chain complex of X is related to the chain complexes of A, B and $A \cap B$ via a short exact sequence of chain complexes.

(b) Using a long exact homology sequence, calculate the homology groups of the complex X in terms of the homology groups of $A \cap B$ when both A and B are contractible complexes.

Some Solutions

B1: (a). Let $f_n = f^2 \cdot \chi_{(n,\infty)}$. Then $f_n \le f^2 \in L_1(\mathbb{R})$. Also, $f_n \to 0$ point-wisely. By LDCT, $\int_{\mathbb{R}} f_n\,dx = \int_{(n,\infty)} f^2\,dx \longrightarrow 0$ for $n \longrightarrow \infty$.

(b). It is false. Take $f_n(x) = n^{-2}\,(x-n)^{-1/2}$ for $n < x < n+1$ and $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Then $\int f\,dx = 2\sum_{n=1}^{\infty} n^{-2}$, so $f \in L_1(\mathbb{R})$. But $\int_n^{n+1} f^2\,dx = \infty$ for each n.

B2: (a) $|f(x)| \le \int_0^{\pi} |x(t)|dt = ||x||_1$. Hence $||f|| \le 1$. Take $x_n = n \cdot \chi_{(0,1/n)}$. Then $||x_n||_1 = 1$ and $|f(x_n)| \to 1$. Therefore, $||f|| = 1$.

(b). By Hölder's inequality, $|f(x)| \le ||x||_2 \cdot (\int_0^{\pi} \cos^2 t\,dt)^{1/2} = \sqrt{\pi/2} \cdot ||x||_2$. The inequality is sharp (take $x(t) = \cos t$). Therefore, $||f|| = \sqrt{\pi/2}$.

(c). In the space $C[0,\pi]$ we have $|f(x)| \le \int_0^{\pi} |\cos t|\,dt \cdot ||x||_{\infty}$, so $||f|| \le 2$. The value 2 is achieved by $|f(x_n)|$ when $x_n(t) = 1$ on $[0, \pi/2 - 1/n]$ and $x_n(t) = -1$ on $[\pi/2 + 1/n, \pi]$ with x_n linear on $[\pi/2 - 1/n, \pi/2 + 1/n]$. Since x_n is continuous and $||x||_{\infty} = 1$, we have $||f|| = 2$.

B3: If f has infinitely many zeros $\{a_n\}$ in $\mathbb{D}$, then we let $z_n = a_n$.

Next consider the case f has no zeros in $\mathbb{D}$. Then on each $\overline{D(0, 1-1/n)}$, by the maximum modulus theorem applied to $1/f$, $|f|$ attains its minimum on the boundary of this disc. Thus there are $z_n = (1-1/n)e^{i\theta_n}$ such that $|f(0)| \ge |f(z_n)|$. Clearly $\{f(z_n)\}$ is bounded and $|z_n| \to 1$.

Finally, if f has finitely many zeros $c_1, \ldots, c_k$ counted with multiplicity in $\mathbb{D}$, then $f(z) = (z-c_1)\cdots(z-c_k)g(z)$, where g is holomorphic and has no zeros in $\mathbb{D}$. By the previous case, we find $\{z_n\}$ satifying the conditions of the question for g. The same $\{z_n\}$ works for f too.

B4: Let γ be $\partial\mathbb{D}$ traced once in the counterclockwise direction, and let $\Gamma = h \circ \gamma$; so $\gamma(t) = e^{it}$ and $\Gamma(t) = h(e^{it})$ for $0 \le t \le 2\pi$. By the argument principle, the winding number of Γ around 0 is N, and the net increase in the argument of $\Gamma(t)$ from $t = 0$ to $t = 2\pi$ is $2\pi N$. Letting $\Gamma(t) = R(t)e^{i\Theta(t)}$, this means that $\Theta(2\pi) - \Theta(0) = 2\pi N$, where Θ is not one fixed branch of the argument but rather several branches joined and to end. But $\mathrm{Re}(h(e^{it})) = R(e^{it})\cos(\Theta(t))$, and Θ changes through exactly N periods of the cosine. The cosine has at least two zeros in each of its periods. Then $\mathrm{Re}(h)$ has at least $2N$ zeros on γ.

D1: The key steps are as follows. By Segre embedding we can consider the case where one of the curves is a hyperplane. The complement of a hyperplane is affine. If the two curves do not intersect, then one of them, living in the complement of a hyperplane is affine. But the only projective variety which is at the same time affine is a point. This contradiction shows that these two curves must intersect.

The algebraic closure of k is used in showing that the global regular functions of a projective variety are only the constants. This then is used to show that if a projective variety is affine, its ring of global regular functions being constants forces it to be a point as an affine variety.

A typical example over the reals would be the parabola $zy = x^2$ and the line $y = x - z$. When $z = 1$ they certainly do not intersect as their graphs would suggest. Moreover the parabola intersects the line at infinity at the point $[0:1:0]$ where as the line intersects it at the point $[1:1:0]$. Hence they do not intersect at all in $\mathbb{P}^2_{\mathbb{R}}$.

D2: Let ϕ be a regular function on $\mathbb{P}^n - (H_i \cap H_j)$. Restricting ϕ to $\mathbb{P}^n - H_i$ we get a regular function there. But regular functions on $\mathbb{P}^n - H_i$ are of the form $\dfrac{f}{x_i^m}$ where f is a homogeneous polynomial in $x_0, \ldots, x_n$ of degree $m \geq 0$, and $x_i \not| f$. Similarly restricting ϕ to $\mathbb{P}^n - H_j$ we get a regular function of the form $\dfrac{g}{x_j^r}$ where g is homogeneous of degree $r \geq 0$ and $x_j \not| g$. On the intersection $(\mathbb{P}^n - H_i) \cap (\mathbb{P}^n - H_j)$ we must have $\dfrac{f}{x_i^m} = \dfrac{g}{x_j^r}$. This gives $fx_j^r = gx_i^m$. But since $i \neq j$, x_j does not divide the right hand side. This forces $r = 0$. Similarly $m = 0$. So $f = g$ are homogeneous of degree 0, i.e. they are constants. On the other hand ϕ, being a constant on an open set, is constant throughout.