BILKENT UNIVERSITY
PhD PROGRAMME
QUALIFYING EXAM
IN MATHEMATICS

11 January 2016

Instructions:

• The FOUR sections are labelled A, B, C, D. Attempt at most TWO questions from each of the four sections A, B, C, D. Thus, you are to attempt at most EIGHT questions altogether.

• Hand in separate scripts for each examiner.

Examiner 1: Algebra, questions A1, A2, A3.

Examiner 2: Real Analysis, questions B1, B2.

Examiner 3: Complex Analysis, questions B3, B4.

Examiner 4: Methods of Applied Mathematics, questions C1, C2, C3, C4.

Examiner 5: Topology, questions D1, D2.

Time allowed: three hours.

Section A: Algebra

A1: Let E be the splitting field of the polynomial $X^4 - X^2 - 1$.
(1) Show that $|E : \mathbb{Q}| = 8$.
(2) What is the Galois group of the extension $E/\mathbb{Q}$?
(3) How many fields K are there such that $\mathbb{Q} \leq K \leq E$ and the extension $K/\mathbb{Q}$ is normal?

A2: Let n be a positive integer and let $T_n(\mathbb{C})$ be the algebra of upper triangular $n \times n$ matrices over $\mathbb{C}$.
(a) Up to isomorphism, how many simple $T_n(\mathbb{C})$-modules are there?
(b) Show that $T_n(\mathbb{C})$ has at least $n(n+1)/2$ isomorphism classes of indecomposable modules.

A3: The group $\mathrm{SL}_2(3)$ has order 24, it has a normal subgroup isomorphic to Q_8, and it has conjugacy classes of sizes 1, 1, 6, 4, 4, 4, 4 consisting of elements of orders 1, 2, 4, 3, 3, 6, 6, respectively.
(a) Let G be a finite group, let a be a non-central element of G with order 3 and let χ be the $\mathbb{C}G$-character of a faithful 2-dimensional $\mathbb{C}G$-module V. By considering the eigenvalues of the action of a on V, find the three possible values of $\chi(a)$.
(b) Hence or otherwise, construct the ordinary character table of $\mathrm{SL}_2(3)$.

Section B: Analysis

B1: Show that the Volterra operator $V : L_2(0,1) \longrightarrow L_2(0,1) : x \mapsto \int_0^t x(s)\,ds$ is not normal. (Recall that a bounded operator $A : H \longrightarrow H$ is normal if $AA^* = A^*A$, where H is a Hilbert space, A^* is the adjoint of A.)

B2: Suppose $f \in L(X,\ \mathcal{X},\ \mu)$ and $\int_E f\,d\mu = 0$ for any measurable set E. Show that $f \overset{ae}{=} 0$.

B3: Let $V = \{\, z \in \mathbb{C} : |z| > 4 \,\}$. Show that there is a holomorphic F on V such that

$$F'(z) = \frac{z}{(z-1)(z-2)(z-3)}$$

but also show that there is no holomorphic G on V such that

$$G'(z) = \frac{z^2}{(z-1)(z-2)(z-3)}\ .$$

B4: Suppose $\{f_n\}$ is a sequence of holomorphic functions on the unit disc $\mathbb{D}$ such that $f_n(\mathbb{D}) \subset \mathbb{D}$. If there is one $a \in \mathbb{D}$ such that $|f_n(a)| \to 1$ as $n \to \infty$, prove that then $|f_n(z)| \to 1$ as $n \to \infty$ for all $z \in \mathbb{D}$.

Section C: Applied Mathematics

C1: Let $z = z_0$ be a regular singular point of the differential equation $u'' + p(z)\, u' + q(z)\, u = 0$ with indices r_1 and r_2. Show that the solution of this differential equation about $z = z_0$ corresponding to the index r_2 has the form

$$u(z) = A\, u_1(z)\, \ln(z - z_0) + (z - z_0)^{r_2} \sum_{n=0}^{\infty} C_n\, (z - z_0)^n$$

when $r_1 - r_2$ is a positive integer or zero. Here A and C_n's are constants. (Hint: If u_1 is given then $u_2 = C u_1 \displaystyle\int^x \frac{dx'}{u_1^2\,(x')}\, e^{-\int^{x'} \frac{b(\xi)}{a(\xi)}\, d\xi}$ where C is a constant.)

C2: Find the series solution of the equation $u'' - z\, u' - u = 0$ about the point $z = 0$ with $u(0) = \alpha$ and $u'(0) = \beta$ where α and β are some given constants.

C3: Let $u, v \in C^2(D)$ be twice differentiable functions, where $D \subset R^2$. Let

$$J(u, v) = \int\int_D [(u_x)^2 + (u_t)v + (v_t)^2] dt dx$$

with the initial and boundary conditions (i) $u(0, x) = 2v_t(0, x)$, (ii) $u(t, 0) = 0$, $u(t, \pi) = \varepsilon \sin(t/2\varepsilon)$, (iii) $v(t, 0) = 0$. Solve this variational problem. Is this problem well posed? Prove your statement (consider the limit $\varepsilon \to 0$).

C4: Let $u_{tt} + k\, u_t = c^2\, u_{xx} + F(x, t)$ for $0 < x < L,\ t > 0$ with

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L,$$

$$u(0, t) = r(t), \quad u(L, t) = s(t), \quad t \geq 0$$

where $f(x), g(x), r(t)$ and $s(t)$ are given functions, k and c are constants, $F(x, t)$ is a given function of x and t. Prove that:

(i) $f(0) = r(0), f(L) = s(0), g(0) = r'(0)$ and $g(L) = s'(L)$.

(ii) Assume that there exists a solution of the above problem. Prove that the solution is unique. (Hint: use the energy functional $E(t) = \int_0^L [(u_t)^2 + c^2\, (u_x)^2] dx$.)

Section D: Geometry and Topology

D1: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. If there exist (a, b, c) and (g, h, i) in $\mathbb{R}^3$ such that $f(a, b, c) > 0$ and $f(g, h, i) < 0$ then show that the set $Z(f) = \{(x, y, z) \in \mathbb{R}^3 \,|\, f(x, y, z) = 0\}$ is uncountable.

D2: A subset A of a topological space X is called a retract of X if there exists a continuous function $r : X \to A$ (called a retraction) such that $r(a) = a$ for all $a \in A$. Moreover A is called a deformation retract of X if there exists a continuous function $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) = r(x)$ for all $x \in X$ and $H(a, t) = a$ for all $a \in A$ and $t \in I$. Let $\mathbb{S}^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$. Show that $A = \mathbb{S}^1 \times \{1\}$ is a retract of $X = \mathbb{S}^1 \times \mathbb{S}^1$ however it is not a deformation retract of X.

Some Solutions

B3: Let first γ be the circle in V of radius 5 centered at 0 traced once in the counterclockwise direction. Let $g(z)$ be the right-hand side of the equation for $G'(z)$. Then

$$\begin{aligned}\int_\gamma g(z)\,dz &= 2\pi i\big(\mathrm{Res}(g,1)+\mathrm{Res}(g,2)+\mathrm{Res}(g,3)\big)\\ &= 2\pi i\left(\frac{z^2}{(z-2)(z-3)}\bigg|_{z=1}+\frac{z^2}{(z-1)(z-3)}\bigg|_{z=2}+\frac{z^2}{(z-1)(z-2)}\bigg|_{z=3}\right)\\ &= 2\pi i\left(\frac{1}{2}-4+\frac{9}{2}\right)\neq 0.\end{aligned}$$

Thus g has no primitive on V.

For F, we have to consider all closed curves γ in V. Let $f(z)$ be the right-hand side of the equation for $F'(z)$. If γ is a closed curve in V not surrounding the hole of V, then its winding number around each of $1,2,3$ is 0 and $\int_\gamma f(z)\,dz = 0$. Next let γ be a closed curve in V surrounding the hole of V. Then the winding number of γ around each of $1,2,3$ is the same; let it be n. Then

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\pi i n\big(\mathrm{Res}(f,1)+\mathrm{Res}(f,2)+\mathrm{Res}(f,3)\big)\\ &= 2\pi i n\left(\frac{z}{(z-2)(z-3)}\bigg|_{z=1}+\frac{z}{(z-1)(z-3)}\bigg|_{z=2}+\frac{z}{(z-1)(z-2)}\bigg|_{z=3}\right)\\ &= 2\pi i n\left(\frac{1}{2}-2+\frac{3}{2}\right)= 0.\end{aligned}$$

Thus $\int_\gamma f(z)\,dz = 0$ for all closed curves γ in V and f has a primitive on V. $\square$

B4: Suppose that there is a $b\in\mathbb{D}$ such that $|f_n(b)|\not\to 1$. Since $\{f_n(b)\}$ is bounded, it has a subsequence $\{f_{n_k}(b)\}$ converging to c with $|c|<1$. Since $\{f_{n_k}\}$ is uniformly bounded, it is a normal family. Then by the Montel theorem, it has a further subsequence $\{f_{n_{k_m}}\}$ converging uniformly on compact subsets of $\mathbb{D}$ to a holomorphic f on $\mathbb{D}$. Clearly $|f|\le 1$ on $\mathbb{D}$. But $|f(a)|=1$ and then $a\in\mathbb{D}$ is a local maximum for f. By the maximum modulus theorem, f is a constant of modulus 1. This contradicts that $|f(b)|=|c|<1$.